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MYRTLE CLEMENS

NEW

SCHOOL ALGEBRA

BY

G. A. WENTWORTH

AUTHOR OF A SERIES OF TEXT-BOOKS IN MATHEMATICS

GINN & COMPANY

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PREFACE.

The first chapter of this book prepares the way for quite a full treatment of simple integral equations with one unknown number. In the first two chapters only *positive* numbers are involved, and the beginner is led to see the practical advantages of Algebra before he encounters the difficulties of negative numbers.

The definitions and explanations contained in these chapters should be carefully *read* at first; after the learner has become familiar with algebraic operations, special attention should be given to the principal definitions.

The third chapter contains a simple explanation of negative numbers. The recognition of the fact that the real nature of subtraction is counting backwards, and that the real nature of multiplication is forming the product from the multiplicand precisely as the multiplier is formed from unity, makes an easy road to the laws of addition and subtraction of algebraic numbers, and to the law of signs in multiplication and division. All the principles and rules of this chapter are illustrated and enforced by numerous examples involving *simple* algebraic expressions only.

The ordinary processes with *compound* expressions, including cases of resolution into factors, and the treatment of fractions, naturally follow the third chapter. The immediate succession of topics that require similar work is of the highest importance to the beginner, and it is hoped that the chapters on compound expressions will prove interesting, and give sufficient readiness in the use of symbols.

The chapter on Factors has been made as complete as possible for an elementary text-book, with a view to shorten subsequent work. The easy method of resolving quadratic trinomials into factors, whether the coefficient of the square of the letter involved is unity or greater than unity, and the Factor Theorem, explained on page 102, will be found of very great service in abridging algebraic processes. Examples of short methods for finding the highest common factor of compound expressions are given on page 118; and examples of short methods for solving quadratic equations by resolving them into factors are given on pages 272 and 273.

A five-place table of logarithms is placed at the end of the book instead of a four-place table. Five-place logarithms are in common use for practical calculations, and are required by most colleges and science schools for the solution of problems set in entrance examination papers.

The exercises throughout the book are carefully graded. They are sufficiently varied and interesting, and are not so difficult as to discourage the learner, or so easy as to deprive him of the satisfaction of well-earned success.

The author has spared no pains to make this a model text-book in subject-matter and mechanical execution. The remarkable favor with which his other Algebras have been received is shown by the fact that nearly a million copies have already been sold, and the sale continues to increase from year to year. The author trusts that this new candidate for favor will have the same generous reception, and be found to meet fully the requirements of the recent advance in the science and method of teaching Elementary Algebra.

The author is under obligations to many teachers for valuable suggestions, and he will be thankful for corrections or criticisms.

G. A. WENTWORTH.

Exeter, N. H., June, 1898

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NOTICE TO TEACHERS.

Pamphlets containing the answers will be furnished without charge to teachers for their classes, *on application to GINN & COMPANY, Publishers.*

NEW SCHOOL ALGEBRA.

CHAPTER I.

DEFINITIONS AND NOTATION.

Numbers and Number-Symbols.

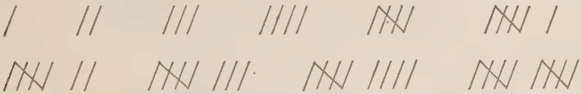
1. Algebra. Algebra, like Arithmetic, treats of numbers.

2. Units. In counting separate objects or in measuring magnitudes, the *standards* by which we count or measure are called **units.**

Thus, in counting the boys in a school, the unit is a boy; in selling eggs by the dozen, the unit is a dozen eggs; in selling bricks by the thousand, the unit is a thousand bricks; in expressing the measure of short distances, the unit is an inch, a foot, or a yard; in expressing the measure of long distances, the unit is a rod, or a mile.

3. Numbers. *Repetitions of the unit* are expressed by *numbers.*

A single unit and groups of units formed by successive additions of a unit may be represented as follows:



These representative groups are named one, two, three, four, five, six, seven, eight, nine, ten; and are known collectively under the general name of **numbers.** It is obvious that these representative groups will have the same meaning, whatever the units may be that are counted.

4. Quantities. A number of specified units of any kind is called a *quantity;* as 4 pounds, 5 oranges.

NOTE. Quantities are often called *concrete numbers*, the adjective *concrete* being transferred from the units counted to the numbers that count them; but a number signifies the *times* a unit is taken, whether the unit is expressed or understood, and *is always abstract.*

Thus, 4 barrels of flour means 4 times 1 barrel of flour; and 10 cords of wood means 10 times 1 cord of wood.

5. Number-Symbols in Arithmetic. Instead of groups of straight marks, we use in Arithmetic the arbitrary symbols 1, 2, 3, 4, 5, 6, 7, 8, 9, called **Arabic numerals,** for the numbers one, two, three, four, five, six, seven, eight, nine.

The next number, ten, is indicated by writing the figure 1 in a different position, so that it shall signify not *one*, but *ten*. This change of position is effected by introducing a new symbol, 0, called **nought** or **zero,** and signifying *none*.

All succeeding numbers up to the number consisting of 10 tens are expressed by writing the figure for the number of tens they contain in the second place from the right, and the figure for the number of units besides in the first place. The *hundreds* of a number are written in the *third place* from the right. The *thousands* are written in the *fourth place* from the right; and so on.

6. Number-Symbols in Algebra. Algebra employs the *letters of the alphabet* in addition to the figures of Arithmetic to represent numbers. The letters of the alphabet are used as *general* symbols of numbers to which *any particular values* may be assigned. In any problem, however, a letter is understood to have the same value throughout the problem.

7. Terms Common to Arithmetic and Algebra. Terms common to Arithmetic and Algebra, as addition, sum, subtraction, minuend, subtrahend, difference, etc., have the same meaning in both; or an *extended meaning* in Algebra consistent with the sense attached to them in Arithmetic.

The Principal Signs of Operations.

The principal signs of operations are the same in Algebra as in Arithmetic.

8. The Sign of Addition, +. The sign + is read *plus.*

Thus, $4 + 3$ is read 4 plus 3, and indicates that the number 3 is to be added to the number 4; $a + b$ is read a plus b, and indicates that the number b is to be added to the number a.

9. The Sign of Subtraction, −. The sign − is read *minus.*

Thus, $4 - 3$ is read 4 minus 3, and indicates that the number 3 is to be subtracted from the number 4; $a - b$ is read a minus b, and indicates that the number b is to be subtracted from the number a.

10. The Sign of Multiplication, × The sign × is read *times,* or *multiplied by.*

Thus, 4×3 is read 4 times 3, and indicates that the number 3 is to be multiplied by 4; $a \times b$ is read a times b, and indicates that the number b is to be multiplied by the number a.

A dot is sometimes used for the sign of multiplication. Thus, $2 \cdot 3 \cdot 4 \cdot 5$ means the same as $2 \times 3 \times 4 \times 5$. Either sign is read *multiplied by* when followed by the multiplier. $\$a \times b$, or $\$a \cdot b$, is read a dollars multiplied by b.

11. The Sign of Division, ÷. The sign ÷ is read *divided by.*

Thus, $4 \div 2$ is read 4 divided by 2, and indicates that the number 4 is to be divided by 2; $a \div b$ is read a divided by b, and indicates that the number a is to be divided by the number b.

Division is also indicated by writing the dividend above the divisor with a horizontal line between them; or by separating the dividend from the divisor by an oblique line, called the solidus.

Thus, $\frac{a}{b}$, or a/b, means the same as $a \div b$.

NOTE. The operation of adding b to a, of subtracting b from a, of multiplying a by b, or of dividing a by b is *algebraically complete* when the two letters are connected by the proper sign.

12. The Radical Sign, $\sqrt{}$. The sign $\sqrt{}$ is called the *radical sign,* and denotes that a root of the number before which it is placed is to be found.

Other Signs Used in Algebra.

13. The Sign of Equality, $=$. The sign $=$ is read *is equal to,* and when placed between two numbers indicates that these two numbers are *equal.*

Thus, $8 + 4 = 12$ means that the sum of 8 and 4 is equal to 12; $x + y = 20$ means that the sum of x and y is equal to 20; and $x = a + b$ means that x is equal to the sum of a and b.

14. The Sign of Deduction, $\therefore$. The sign $\therefore$ is read *hence* or *therefore.*

15. The Sign of Continuation, ····· The sign ····· is read *and so on.*

Thus, 1, 2, 3, 4, ····· is read one, two, three, four, *and so on.* a_1, a_2, a_3, ····· a_n is read a sub one, a sub two, a sub three, *and so on to* a sub n. a', a'', a''', ····· is read a prime, a second, a third, *and so on.*

16. The Signs of Aggregation. The signs of aggregation are the parenthesis ( ), the bracket [ ], the brace { }, the vinculum —, and the bar |.

These signs mean that the indicated operations in the expressions affected by them are to be performed first, and the result treated as a single number.

Thus, $(a + b) \times c$ means that the *sum* of a and b is to be multiplied by c; $(a - b) \times c$ means that the *difference* of a and b is to be multiplied by c.

The vinculum is written over the expression that is to be treated as a single number.

Thus, $a - \overline{b + c}$ means the same as $a - (b + c)$, and signifies that the sum of b and c is to be subtracted from a; and $\sqrt{a - b}$ means the same as $\sqrt{}(a - b)$, and signifies that b is to be subtracted from a, and the square root of the remainder found.

Factors, Powers, Roots.

17. Factors. When a number is the product of two or more numbers, each of these numbers, or the product of two or more of them, is called a **factor** of the given number.

Thus, 2, a, b, $2a$, $2b$, ab are factors of $2ab$.

18. Factors expressed by letters are called **literal** factors; factors expressed by figures are called **numerical** factors.

19. The sign × is omitted between factors, if the factors are letters, or a numerical factor and a literal factor.

Thus, we write $63ab$ for $63 \times a \times b$; we write abc for $a \times b \times c$.

20. The expression abc must not be confounded with $a + b + c$. abc is a product; $a + b + c$ is a sum.

If $a = 2,\ b = 3,\ c = 4,$

then $abc = 2 \times 3 \times 4 = 24;$

but $a + b + c = 2 + 3 + 4 = 9.$

NOTE. When a sign of operation is omitted in the notation of Arithmetic, it is always the *sign of addition;* but when a sign of operation is omitted in the notation of Algebra, it is always the *sign of multiplication.* Thus, 456 means $400 + 50 + 6$, but $4ab$ means $4 \times a \times b$.

21. If one factor of a product is equal to 0, the product is equal to 0, whatever the values of the other factors. Such a factor is called a **zero factor.**

Thus, $abcd = 0$, if a, b, c, or $d = 0$.

22. Coefficients. Any factor of a product may be considered as the **coefficient** of the remaining factors; that is, the **co-factor** of the remaining factors. Coefficients expressed by letters are called *literal* coefficients; expressed by Arabic numerals, *numerical* coefficients.

Thus, in $7x$, 7 is the *numerical coefficient* of x; in ax, a is the *literal coefficient* of x.

If no numerical coefficient is written, 1 is understood.

23. Powers and Roots. When a number is taken a number of times as a *factor*, the result is called a **power** of the factor. When a number is the product of *equal factors*, one of the equal factors is called a **root** of the number.

24. Indices or Exponents of Powers. An index or exponent of a power is a number-symbol written at the right of and a little above the number.

If the exponent is a *whole number*, it shows the number of times the given number is taken as a factor.

Thus, a^1, or simply a, denotes that a is taken *once* as a factor; a^2 denotes that a is taken *twice* as a factor; a^3 denotes that a is taken *three times* as a factor; and so on. These are read: the first power of a; the second power of a; the third power of a; and so on. We write a^3 for aaa, a^4 for $aaaa$, a^n for $aaaa \cdots$ to n factors.

NOTE. The second power of a number is often called the *square* of that number; thus, a^2 is called the *square* of a, because if a denotes the number of units of length in the side of a square, a^2 denotes the number of units of surface in the square. The third power of a number is often called the *cube* of that number; thus, a^3 is called the *cube* of a, because if a denotes the number of units of length in the edge of a cube, a^3 denotes the number of units of volume in the cube.

25. The meaning of coefficient and exponent must be carefully distinguished. Thus,

$$4a = a + a + a + a;$$
$$a^4 = a \times a \times a \times a.$$

If $a = 3$,
$$4a = 3 + 3 + 3 + 3 = 12.$$
$$a^4 = 3 \times 3 \times 3 \times 3 = 81.$$

26. Indices of Roots. An index of a root is a number-symbol written above the radical sign to indicate the required root.

Thus, $\sqrt[2]{a}$, or simply $\sqrt{a}$, means one of the *two* equal factors of a, that is, the square root of a; $\sqrt[3]{a}$ means one of the *three* equal factors of a, that is, the cube root of a; and so on.

EXERCISE 1.

If $a = 1$, $b = 2$, $c = 3$, $d = 4$, $x = 5$, $y = 6$, $z = 0$, find the numerical value of:

1. $15x$.
2. $3ab$.
3. $7by$.
4. $5bd$.
5. $9y^2$.
6. $3b^2c$.
7. $4c^2x^2$.
8. $2b^5x$.
9. b^2cy^2.
10. $a^4b^3c^2$.
11. $\frac{3}{4}b^2c^2$.
12. $\frac{2}{3}c^2y^2$.
13. $\frac{3}{8}b^2y^2$.
14. $\frac{2}{5}xy^3$.
15. $\frac{5}{7}x^4z^5$.
16. $\frac{5}{12}c^3d^3$.
17. $\frac{3}{16}d^2x^3$.
18. $\frac{3}{7}a^4z^2$.
19. $\frac{5}{2}b^2y^2$.
20. $\frac{7}{3}c^2d^2$.
21. $\dfrac{3a^2d^2}{8bc}$.
22. $\dfrac{7b^2y}{2dc}$.
23. $\dfrac{x^2y^2}{5b^2c^2}$.
24. $\dfrac{x^2y^2z^2}{3c^3d^3}$.
25. $\dfrac{10a^5b^5}{d^bb^a}$.
26. $\sqrt{4y}$.
27. $\sqrt{dx}$.
28. $\sqrt[3]{8c}$.
29. $\sqrt[3]{bdx^3}$.
30. $\sqrt{(bcy)}$.
31. $\sqrt{ab^2x^2}$.
32. $\sqrt{x^3y^3z^3}$.
33. $\sqrt[3]{3bc^2d}$.
34. $2\sqrt{b^2dc^2}$.
35. $c\sqrt{dx^2}$.

Numerical Values of Compound Expressions.

36. Each term should be written in the algebraic form by omitting the sign × between a numerical factor and a literal factor or between two literal factors. The operations indicated in a term must be performed *before* the operation indicated by the sign prefixed to the term.

37. The **parts of a term** are combined in the order of the signs × and ÷ *from left to right.*

The **terms** of an expression are combined in the order of the signs + and − *from left to right.*

Thus, $60 - 40 \div 5 \times 3 - 20 = 60 - \frac{40}{5} \times 3 - 20 = 16$.

38. The terms may be arranged in any order before combining them. This is called the **commutative law** for addition and subtraction.

Numerical Values of Compound Expressions.

1. If $b = 10$, $c = 2$, $y = 5$, find the numerical value of $6b - (8y \div 2c)c - 2cy$.

$$6b - (8y \div 2c)\,c - 2cy = 6 \times 10 - \tfrac{40}{4} \times 2 - 2 \times 2 \times 5$$
$$= 60 - 20 - 20 = 20.$$

2. If $x = 7$, $y = 5$, find the numerical value of

$$(x + y)(x - y) + \frac{x + y}{x - y}.$$

$$(x + y)(x - y) + \frac{x + y}{x - y} = (7 + 5)(7 - 5) + \frac{7 + 5}{7 - 5}$$
$$= 12 \times 2 + \tfrac{12}{2} = 30.$$

Exercise 2.

If $a = 1$, $b = 2$, $c = 3$, find the value of:

1. $7a - bc$.
2. $ac + b$.
3. $4ab - c$.
4. $6ab - b - c$.
5. $2a - b + c$.
6. $ab + bc - ac$.
7. $b^2 + a^2 + c^2$.
8. $5bc^2 - 2ab$.
9. $\sqrt{4abc} + 2c$.
10. $\sqrt{6abc} - 2b$.
11. $c^3 - b^3$.
12. $\sqrt[3]{c^2 - a^3}$.
13. $2b + 3c - a$.
14. $(a + b)^2 + 2(c - a)^2$.
15. $\sqrt{6bc} - b - c$.
16. $6b - 10bc \div 12a + 2c$.
17. $5c \div (b - a) - b - a$.
18. $\sqrt[3]{6b^2c^2} - \sqrt{b^2c^2}$.

If $a = 1$, $b = 2$, $c = 3$, $d = 0$, find the value of:

19. $7a - bc + 6d$.
20. $ac + b - d$.
21. $4ab - cd - d$.
22. $2a - b + cd$.
23. $ab + bc - ad$.
24. $2ab - 5bcd$.
25. $\sqrt{4abcd} + 2b^2$.
26. $\sqrt{4abcd} + b^c$.
27. $b^3 - c + d^c$.
28. $3c^3 - 2b^2 + a$.
29. $2b + (5c - 3) \div (2c - a)$.
30. $3c^2 - 2a^a - 2b^b$.

Parentheses.

39. A parenthesis preceded by the sign +. If a man has 10 dollars and afterwards collects 3 dollars and then 2 dollars, it makes no difference whether he puts the 3 dollars and the 2 dollars together and adds their sum to his 10 dollars, or adds the 3 dollars to his 10 dollars, and then the 2 dollars.

The first process is represented by $10 + (3 + 2)$.
The second process is represented by $10 + 3 + 2$.

Hence, $$10 + (3 + 2) = 10 + 3 + 2. \qquad (1)$$

If a man has 10 dollars and afterwards collects 3 dollars and then pays a bill of 2 dollars, it makes no difference whether he pays the 2 dollars from the 3 dollars collected and adds the remainder to his 10 dollars, or adds the 3 dollars collected to his 10 dollars and pays from this sum his bill of 2 dollars.

The first process is represented by $10 + (3 - 2)$.
The second process is represented by $10 + 3 - 2$.

Hence, $$10 + (3 - 2) = 10 + 3 - 2. \qquad (2)$$

If we use general symbols in (1) and (2), we have,

$$a + (b + c) = a + b + c,$$

and $$a + (b - c) = a + b - c.$$ Hence,

We have the general rule for a parenthesis preceded by +:

If an expression within a parenthesis is preceded by the sign +, the parenthesis may be removed without making any change in the signs of the terms of the expression.

Instead of a parenthesis, any other sign of aggregation may be used and the same rule will apply.

40. A parenthesis preceded by the sign —. If a man with 10 dollars has to pay two bills, one of 3 dollars and one of 2 dollars, it makes no difference whether he takes 3 dollars and 2 dollars at one time, or takes 3 dollars and 2 dollars in succession, from his 10 dollars.

The first process is represented by $10 - (3 + 2)$.
The second process is represented by $10 - 3 - 2$.

Hence, $$10 - (3 + 2) = 10 - 3 - 2. \tag{1}$$

If a man has 10 dollars consisting of two 5-dollar bills, and has a debt of 3 dollars to pay, he can pay his debt by giving a 5-dollar bill and receiving 2 dollars.

This process is represented by $10 - 5 + 2$.

Since the debt paid is 3 dollars, that is, $(5 - 2)$ dollars, the number of dollars he has left can be expressed by

$$10 - (5 - 2).$$

Hence, $$10 - (5 - 2) = 10 - 5 + 2. \tag{2}$$

If we use general symbols in (1) and (2), we have,

$$a - (b + c) = a - b - c,$$

and $$a - (b - c) = a - b + c.$$ Hence,

We have the general rule for a parenthesis preceded by — :

If an expression within a parenthesis is preceded by the sign —, the parenthesis may be removed, provided the sign before each term within the parenthesis is changed, the sign + to — and the sign — to +.

NOTE. If the *vinculum* is used, the sign prefixed to the first term under the vinculum must be understood as the sign before the vinculum.

Thus, $a + \overline{b - c}$ has the same meaning as $a + (b - c)$,
and $a - \overline{b - c}$ has the same meaning as $a - (b - c)$.

EXERCISE 3.

Remove the parentheses and combine:

1. $9+(3+2)$.
2. $9+(3-2)$.
3. $7+(5+1)$.
4. $7+(5-1)$.
5. $9-(8-6)$.
6. $8-(7-5)$.
7. $9-(6+1)$.
8. $8-(3+2)$.
9. $(3-2)-(2-1)$.
10. $(7-3)-(3-2)$.
11. $(8-2)-(5-3)$.
12. $15-(10-3-2)$.

If $a=10$, $b=5$, $c=4$, $d=2$, find the value of:

13. $(a+b)+(c+d)$.
14. $(a+b)-(c-d)$.
15. $(a-b)-(c-d)$.
16. $(a-b)+(c-d)$.

Product of a Compound by a Simple Factor.

41. In finding the product of $4\,(5+3)$, it makes no difference in the result whether we multiply the sum of 5 and 3 by 4, or multiply 5 by 4 and 3 by 4 and add the products.

By the first process, $4\,(5+3)=4\times 8=32$.

By the second process, $4\,(5+3)=(4\times 5+4\times 3)=32$.

In like manner, $4\,(5-3)=4\times 2=\ 8$,

and $4\,(5-3)=(4\times 5-4\times 3)=\ 8$.

In general symbols, $a\,(b+c)=ab+ac$,

and $a\,(b-c)=ab-ac$.

This is called the **distributive law** for multiplication.

42. *The order of the factors is immaterial.*

Thus, $4\,(5+3)=4\times 5+4\times 3=32$,

and $(5+3)\,4=5\times 4+3\times 4=32$.

In general symbols, $ab=ba$.

This is called the **commutative law** for multiplication.

Perform the indicated operations:

1. $x+3\,(a-b)$.
2. $x-3\,(a-b)$.

1. $x+3\,(a-b)=x+(3\,a-3\,b)=x+3\,a-3\,b$.
2. $x-3\,(a-b)=x-(3\,a-3\,b)=x-3\,a+3\,b$.

EXERCISE 4.

Perform the indicated operations, and find the numerical value of each expression, if $a = 5$, $b = 4$, $c = 3$:

1. $3(ab + c)$.
2. $4(ac + b)$.
3. $2(ab - c)$.
4. $5(ac - b)$.
5. $7(bc - a)$.
6. $ac(b - c)$.
7. $bc + a(b - c)$.
8. $b + 2(a - c)$.
9. $ac - 2(b + c)$.
10. $5ac - 2(b^2 + b)$.
11. $2b + 3(a - c)$.
12. $ab + b(b - c)$.
13. $(a - c)b - a$.
14. $(a - b)a + 2c$.
15. $(b + c)c - 2b$.
16. $(a - c)7 - 3c$.
17. $(a - c)c^2 - 2c^2$.
18. $(a^2 - b^2)c - a^2$.

Quotient of a Compound by a Simple Expression.

43. In finding the quotient of $(8 + 4) \div 2$ it makes no difference in the result whether we divide the sum of 8 and 4 by 2, or divide 8 by 2 and 4 by 2, and add the quotients.

By the first process, $(8 + 4) \div 2 = 12 \div 2 = 6$.

By the second process, $(8 + 4) \div 2 = (8 \div 2 + 4 \div 2) = 6$.

In general symbols, $$\frac{a + b}{c} = \frac{a}{c} + \frac{b}{c},$$

and $$\frac{a - b}{c} = \frac{a}{c} - \frac{b}{c}.$$

This is called the **distributive law** for division.

Perform the indicated operations:

1. $x + (3a + 3b) \div 3$.
2. $x - (3a + 3b) \div 3$.

1. $x + (3a + 3b) \div 3 = x + (a + b) = x + a + b$.
2. $x - (3a + 3b) \div 3 = x - (a + b) = x - a - b$.

EXERCISE 5.

Perform the indicated operations, and find the numerical value of each expression, if $a = 8$, $b = 4$, $c = 2$:

1. $(b + c) \div c$.
2. $(a + c) \div b$.
3. $(a - b) \div b$.
4. $(b - c) \div c$.
5. $(ab + c) \div c$.
6. $(ac + b) \div b$.
7. $(ac - b) \div b$.
8. $(ab - c) \div c$.
9. $(b^2 + c^2) \div a$.
10. $(b^2 - c^2) \div b$.
11. $(a^2 - c^2) \div c^2$.
12. $(a^2 - b^2) \div b^2$.

CHAPTER II.

SIMPLE EQUATIONS.

44. Equations. An equation is a statement in symbols that two expressions stand for the same number.

Thus, the equation $3x + 2 = 8$ states that $3x + 2$ and 8 stand for the same number.

45. That part of the equation which precedes the sign of equality is called the **first member**, or **left side**, and that part of the equation which follows the sign of equality is called the **second member**, or **right side.**

46. An equation containing letters, if true for all values of the letters involved, is called an **identical equation**; but if it is true only for certain particular values of the letters involved, it is called an **equation of condition.**

Thus, $a + b = b + a$, which is true for *all values* of a and b, is an *identical equation;* and $3x + 2 = 8$, which is true only when x stands for 2, is an *equation of condition.*

For brevity, an identical equation is called an **identity,** and an equation of condition is called simply an **equation.**

47. We often employ an equation to discover an *unknown number* from its relation to known numbers. We usually represent the unknown number by one of the *last* letters of the alphabet, as x, y, z; and the known numbers by the *first* letters, a, b, c, and by the Arabic numerals.

48. Simple Equations. Equations which, when reduced to their simplest form, contain only the *first power* of the unknown numbers are called *simple equations*, or *equations of the first degree.*

Thus, $7x + 5 = 4x + 14$, and $ax + b = c$ are simple equations in x.

49. Two or more like terms may be combined to form a single like term by uniting their *numerical coefficients.*

Thus, $3ax + ax = 4ax$; and $5ax - 3ax = 2ax$.

50. To Solve an Equation with One Unknown Number is to find the unknown number; that is, to find the number which, when substituted for its symbol in the given equation, renders the equation an identity.

This number is said to **satisfy** the equation, and is called the **root** of the equation.

51. Axioms. In solving an equation, we make use of the following self-evident truths, called axioms:

Ax. 1. If equal numbers are added to equal numbers, the sums are equal.

Ax. 2. If equal numbers are subtracted from equal numbers, the remainders are equal.

Ax. 3. If equal numbers are multiplied by equal numbers, the products are equal.

Ax. 4. If equal numbers are divided by equal numbers, the quotients are equal.

Ax. 5. If two numbers are equal to the same number, they are equal to each other.

52. Transposition of Terms. It becomes necessary in solving simple equations to bring all the terms that contain the symbols for the unknown numbers to one side of the equation, and all the other terms to the other side. This process is called **transposing the terms.**

1. Find the number for which x stands when

$$x - b = a.$$

Add b to each side, $x - b + b = a + b$. (Ax. 1)

Cancel $-b + b$, $x = a + b$. (§ 34)

The result is the same as if we had transposed $-b$ from the left side to the right side and changed its sign.

2. Find the number for which x stands when

$$x + b = a.$$

Subtract b from each side, $x + b - b = a - b$. (Ax. 2)

Cancel $+b - b$, $x = a - b$. (§ 34)

In this case, we have transposed b from the left side to the right side and changed its sign.

We can proceed in like manner in any other case.

Hence, the general rule:

53. *Any term may be transposed from one side of an equation to the other, provided its sign is changed.*

It follows from axioms 1 and 2 that:

54. *Any term that occurs with the same sign on both sides of an equation may be cancelled.*

If we transpose each term of the equation,

$$c - x = a - b, \qquad (1)$$

we have $b - a = x - c$.

That is, $x - c = b - a$. (2)

Equation (2) is the same as (1) with the sign before each term changed. Hence:

55. *The sign of every term of an equation may be changed without destroying the equality.*

56. Numerical Equations. An equation in which all the known numbers are expressed by Arabic numerals is called a *numerical equation.*

Solution of Simple Numerical Equations in X.

1. Solve $3x - 7 = 14 - 4x.$

Transpose $-4x$ to the left side and -7 to the right side,

$3x + 4x = 14 + 7.$ (§ 53)

Combine, $7x = 21.$ (§ 49)

Divide by 7, $x = 3.$ (Ax. 4)

2. Solve the equation

$$1 - 4(x - 2) = 7x - 3(3x - 1).$$

Multiply the compound factor by the simple factor in each side,

$$1 - (4x - 8) = 7x - (9x - 3).$$

Remove the parenthesis in each side,

$1 - 4x + 8 = 7x - 9x + 3.$ (§ 40)

Transpose, $9x - 4x - 7x = 3 - 1 - 8.$

Change the signs of all the terms,

$4x + 7x - 9x = 1 + 8 - 3.$ (§ 55)

Combine, $2x = 6.$ (§ 49)

Divide by 2, $x = 3.$ (Ax. 4)

57. To Solve a Simple Numerical Equation in x, therefore:

Transpose all the terms that contain x to the left side, and all the other terms to the right side. Combine similar terms, and divide both sides by the coefficient of x.

58. Verification. If the value found for x is substituted for x in the *original equation*, and the equation reduces to an *identity*, the *value* of x, that is, *the root of the equation*, is said to be *verified*.

NOTE. In verifying a solution, as in solving an equation, it is important to notice that the signs of all the terms may be changed.

Show that x stands for 3 in the equation

$$3x - 7 = 14 - 4x.$$

Put 3 for x in this equation, and we have

$$3 \times 3 - 7 = 14 - 4 \times 3,$$

or, $$9 - 7 = 14 - 12,$$

that is, $$2 = 2.$$

EXERCISE 6.

Find the value of x, and verify the answer:

1. $5x - 4 = 16.$
2. $3x + 4 = 25.$
3. $24x - 7x = 34.$
4. $16x = 7x + 81.$
5. $3x = 55 - 2x.$
6. $5x = 3x + 6.$
7. $7x = 6x + 4.$
8. $5x = 28 - 2x.$
9. $2x = 11 + x.$
10. $14x - 79 = 8x - 25$
11. $5x - 4 = 12 - 3x.$
12. $7x + 4 = 3x + 24.$
13. $12x - 16 = 8 + 6x.$
14. $4x - 10 = 14 + 2x.$
15. $2x - 5 = 7 - x.$
16. $4x - 14 = x - 2.$
17. $4x - 11 = 2x - 5.$
18. $4x - 10 = 3x - 5.$
19. $5(x + 1) + 6(x + 2) = 7(x + 3).$
20. $4(x + 7) - 36 = 13(x - 2).$
21. $6(3x - 1) - 8x = 140 + 2(x - 1).$
22. $3(3x - 2) - 6(4 - x) = 24x - 4(7x - 2).$
23. $3(x + 13) - 15 = 4(x - 2) - 9.$
24. $10x - (x - 10) = 3x + 52.$
25. $3x - (x + 5) - (x - 3) = 10 - x.$
26. $x^2 + 8x - (x^2 - x) = 5(x + 3) + 5.$
27. $5 - x + 4(x - 1) - (x - 2) = 15.$
28. $3(x + 10) + 4(x + 20) + 5x = 185 - 3x.$
29. $2(x - 2) + 3(x - 3) + 4(x - 4) = 3x + 7.$
30. $(5x + 3) - 2(x - 1) + (1 - x) = 4(9 - x).$
31. $7 - 21(x + 3) = 13 - 15(2x - 5).$
32. $5(x - 3) - 7(6 - x) + 29 = 50 - 3(8 - x).$

Statement and Solution of Problems.

59. To express in algebraic language the conditions of a problem that are stated in common language is generally very difficult for the beginner. We will therefore give an exercise on translating common language into algebraic language before proceeding to the solutions of problems.

Exercise 7.

1. Write in symbols: a diminished by b; a increased by b; a multiplied by b; a divided by b; the square of a; the square root of a; the cube root of a; the square of a multiplied by the fourth power of b.

2. If a man walks x miles an hour, how many miles will he walk in 4 hours? in a hours?

3. If a man walks 3 miles an hour, how many hours will it take him to walk 12 miles? x miles?

4. If a man walks x miles an hour, how many hours will it take him to walk 20 miles? y miles?

5. What is the divisor, if the dividend is 20 and the quotient 5? if the dividend is a and the quotient b?

6. What is the dividend, if the divisor is 4, the quotient 3, and the remainder 2? if the divisor is d, the quotient q, and the remainder r?

7. What is the quotient, if the dividend is 22, the divisor 4, and the remainder 2? if the dividend is p, the divisor d, and the remainder r?

8. What is the divisor, if the dividend is 22, the quotient 4, and the remainder 2? if the dividend is p, the quotient q, and the remainder r?

9. If one part of 25 is 10, what is the other part?

10. If one part of 30 is x, what is the other part?

11. If one part of x is c, what is the other part?

12. If the sum of two numbers is 40, and one of them is 25, what is the other?

13. If the sum of two numbers is x, and one of them is 5, what is the other?

14. If the sum of two numbers is s, and one of them is a, what is the other?

15. If the difference of two numbers is 7, and the smaller number is 13, what is the greater number?

16. If the difference of two numbers is a, and the smaller number is x, what is the greater number?

17. If the difference of two numbers is c, and the greater number is x, what is the smaller number?

18. Henry is a years old to-day. How old was he 4 years ago? How old will he be in 4 years?

19. John is x years old to-day. How old was he b years ago? How old will he be a years hence?

20. By how much does $5x$ exceed $3x$?

21. By how much does x exceed a?

22. How much does a lack of being x?

23. Write the excess of $2x + 3$ over $x + 1$.

NOTE. If the number to be subtracted is a compound expression it must be enclosed by a parenthesis. Thus, the excess of $2x + 3$ over $x + 1$ is $2x + 3 - (x + 1)$.

24. Write the excess of $3x$ over $4(18 - x)$.

25. Write the excess of $x - 50$ over $80 - x$.

26. What is the excess of $2x - 24$ over $80 - 5x$?

27. What is the excess of $5x + 24$ over $60 - x$?

28. Express in cents a half-dollars and b quarters.

29. Express in cents a dollars b dimes and c cents.

30. A man has a dollars. If he spends b half-dollars and c dimes, how many cents has he left?

31. A man has x dollars y dimes and z cents. If he spends a half-dollars and b quarters, how many cents has he left?

32. A man makes a journey of 236 miles. He travels a miles by train, c miles by boat, and the remainder on foot. How far does he go on foot?

33. A train is running at the rate of a miles an hour. How many miles will it travel in m hours?

34. The floor of a square room measures a feet each way. How many square yards of oilcloth will be required to cover it?

35. The floor of a rectangular room measures a feet by b feet. How many square yards of oilcloth will be required to cover it?

36. A rectangular floor is a feet long and b feet wide. In the middle of the floor there is a square carpet c feet on a side. How many square yards of the floor are bare?

60. In stating problems, x must not be put for money, length, time, weight, etc., but for the required *number* of *specified units* of money, length, time, weight, etc.

Each statement must be made in algebraic symbols, and the meaning of each algebraic statement should be written out in full, in common language.

After the algebraic statements are written, it is necessary and sufficient, in problems involving only one unknown number, to select two expressions that stand for the same number, and to make them the members of the required equation. (Ax. 5.)

Problems Stated and Solved.

1. Three times a certain number is equal to the number increased by 20. Find the number.

Let $x =$ the number.
Then $3x = 3$ times the number;
and $x + 20 =$ the number increased by 20.

But the last two expressions are equal.

Therefore, $3x = x + 20.$
Transposing, $3x - x = 20.$
Combining, $2x = 20.$
Dividing by 2, $x = 10.$

2. John has three times as many oranges as James, and they together have 32. How many has each?

Let x stand for the *number* of oranges James has.
Then $3x$ is the number of oranges John has;
and $x + 3x$ is the number of oranges they together have.

But 32 is the number of oranges they together have.

Therefore, $x + 3x = 32.$
Combining, $4x = 32.$
Dividing by 4, $x = 8.$
Multiplying by 3, $3x = 24.$

Therefore, James has 8 oranges, and John has 24 oranges.

NOTE. Beginners in stating the preceding problem generally write:

Let $x =$ *what* James had.

Now, we know *what* James had. He had oranges, and we are to discover simply the *number* of oranges he had.

3. James and John together have $24, and James has $8 more than John. How many dollars has each?

Let x stand for the number of dollars John has.
Then $x + 8$ is the number of dollars James has;
and $x + (x + 8)$ is the number of dollars they together have.

But 24 is the number of dollars they together have.

Therefore, $x + (x + 8) = 24.$
Removing the parenthesis, $x + x + 8 = 24.$
Combining, $2x = 16.$
Dividing by 2, $x = 8.$
Adding 8 to each side, $x + 8 = 16.$

Therefore, John has \$8, and James has \$16.

Note. The beginner must avoid the mistake of writing

Let $x =$ John's money.

We are required to find the *number* of dollars John has, and therefore x must represent this required number.

4. The sum of two numbers is 18, and three times the greater number exceeds four times the smaller by 5. Find the numbers.

Let $x =$ the greater number.

Then, since 18 is the sum, and x is one of the numbers, the other number must be the sum minus x. Hence,

$18 - x =$ the smaller number.

Now, three times the greater number is $3x$, and four times the less number is $4(18 - x)$; and $3x - 4(18 - x)$ is equal to the excess of three times the greater number over four times the smaller number.

But $5 =$ this excess.

$$\therefore 3x - 4(18 - x) = 5.$$
$$\therefore 3x - (72 - 4x) = 5,$$
or
$$3x - 72 + 4x = 5.$$
$$\therefore 7x = 77,$$
and
$$x = 11.$$

Therefore, the numbers are 11 and 7.

5. A had three times as much money as B. He gave B \$10, and then had only twice as much as B. How much had each at first?

Let $x =$ the number of dollars B had at first.
Then $3x =$ the number of dollars A had at first.

Now, $3x - 10$ is the number of dollars A had after giving \$10 to B, and $x + 10$ is the number of dollars B then had.

Since A's money is twice B's,

$$3x - 10 = 2(x + 10)$$
$$3x - 10 = 2x + 20$$
$$3x - 2x = 20 + 10$$
$$x = 30$$
$$3x = 90.$$

Therefore, B had \$30 and A had \$90.

6. A has \$7 in half-dollars and quarters. If he has 24 coins in all, how many are halves and how many quarters?

Let $x =$ the number of halves.
Then $24 - x =$ the number of quarters.
$50x =$ the value in cents of the halves.
$25(24 - x) =$ the value in cents of the quarters.
Therefore, $50x + 25(24 - x) =$ the value of his money in cents.
But $700 =$ the value of his money in cents.
Hence,

$$50x + 25(24 - x) = 700$$
$$50x + 600 - 25x = 700$$
$$25x = 100$$
$$x = 4$$
$$24 - x = 20.$$

Therefore, he has 4 half-dollars and 20 quarters.

7. A man is now twice as old as his son; 15 years ago he was three times as old. Find the age of each.

Let $x =$ the number of years in the son's age.
Then $2x =$ the number of years in the father's age.
$x - 15 =$ the number of years in the son's age 15 years ago.
$2x - 15 =$ the number of years in the father's age 15 years ago.

But 15 years ago 3 times the son's age was equal to the father's age.

Therefore,

$$3(x - 15) = 2x - 15$$
$$3x - 45 = 2x - 15$$
$$x = 30$$
$$2x = 60.$$

Therefore, the son is 30 years old and the father 60 years old.

Exercise 8.

1. If a number is multiplied by 9, the product is 810. Find the number.

2. If the sum of the ages of a father and son is 56 years, and the father is 7 times as old as the son, what is the age of each?

3. The sum of two numbers is 161, and the greater is 6 times the less. Find the numbers.

4. A tree 100 feet high was broken so that the part broken off was 9 times the length of the part left standing. Find the length of each part.

5. The difference of two numbers is 7, and their sum is 63. Find the numbers.

6. The difference of two numbers is 13, and their sum is 59. Find the numbers.

7. Divide 36 into two parts so that one part shall be greater by 6 than the other part.

8. Three times a given number is equal to the number increased by 36. Find the number.

9. Three times a given number diminished by 20 is equal to the given number. Find the number.

10. One number is 4 times another, and their difference is 24. Find the numbers.

11. The sum of two numbers is 48, and one of them exceeds the other by 6. Find the numbers.

12. The sum of two numbers is 42, and 5 times the smaller number is equal to the larger number. Find the numbers.

13. Find three consecutive numbers, x, $x+1$, and $x+2$, whose sum is 108.

[illegible]

14. Find five consecutive numbers whose sum is 70.

15. A man walks 4 miles an hour for x hours, and another man walks 3 miles an hour for $x + 4$ hours. If they each walk the same distance, how many miles does each walk?

16. A farmer employed two men to build 112 rods of wall. One of them built on the average 4 rods a day, and the other 3 rods a day. How many days did they work?

17. Two men start from the same place and travel in *opposite* directions, one 30 miles a day, and the other 20 miles a day. In how many days will they be 350 miles apart?

18. Two men start from the same place and travel in the *same* direction, one 30 miles a day, and the other 20 miles a day. In how many days will they be 350 miles apart?

19. A man bought 3 equal lots of hay for $255. For the first lot he gave $17 a ton, for the second $16, for the third $18. How many tons of each kind did he buy?

20. A farmer sold a quantity of wood for $84, one half of it at $3 a cord, and the other half at $4 a cord. How many cords of each kind did he sell?

21. If $2x - 3$ stands for 37, for what numb
$7 + x$ stand?

22. At an election two opposing candidates
together 2000 votes, and one received 100 more
the other. How many votes did each candidate

23. If a number is multiplied by 17, the produc
Find the number.

24. The sum of two numbers is 54, and the greater is seventeen times the smaller number. Find the numbers.

25. A tank holding 1500 gallons has three pipes. The first lets in 8 gallons a minute, the second 10 gallons, and

the third 12 gallons a minute. In how many minutes will the tank be filled?

26. The fore and hind wheels of a carriage are 10 feet and 12 feet respectively in circumference. How many feet will the carriage have passed over when the fore wheel has made 100 revolutions more than the hind wheel?

27. Divide a yard of tape into two parts so that one part shall be 6 inches longer than the other part.

28. Divide 23 into two parts such that the sum of twice the greater and three times the smaller is 57.

29. Four times the smaller of two numbers is three times the greater, and their sum is 63. Find the numbers.

30. A farmer sold a sheep, a cow, and a horse for $216. He sold the cow for seven times as much as the sheep, and the horse for four times as much as the cow. How much did he get for each?

31. Distribute $15 among Thomas, Richard, and Henry so that Thomas and Richard shall each have twice as much as Henry.

. Three men, A, B, and C, pay $1000 taxes. B pays
nes as much as A, and C pays as much as A and B
How much does each pay?

hn's age is three times the age of James, and their
her are 16 years. What is the age of each?

vice a certain number increased by 8 is 40. Find
ber.

35. Three times a certain number is 46 more than the number itself. Find the number.

36. One number is four times as large as another. If I take the smaller from 12 and the greater from 21, the remainders are equal. What are the numbers?

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37. Thirty yards of cloth and 20 yards of silk together cost $70; and the silk costs twice as much per yard as the cloth. How much does each cost per yard?

38. In a company of 180 persons, composed of men, women, and children, there are twice as many men as women, and three times as many women as children. How many are there of each?

39. Two trains traveling, one at 25 and the other at 30 miles an hour, start at the same time from two places 330 miles apart, and move toward each other. In how many hours will the trains meet?

40. Twelve persons subscribed for a new boat, but two being unable to pay, each of the others had to pay $4 more than his share. Find the cost of the boat.

41. A tree 84 feet high was broken so that the part broken off was five times the length of the part left standing. Required the length of each part.

42. At an election there were two candidates, and 2800 votes were cast. The successful candidate had a majority of 160. How many votes were cast for each?

43. Divide 20 into two parts such that four times the greater exceeds three times the smaller by 17.

44. The sum of two numbers is 50, and seven times the smaller number exceeds three times the greater number by 10. Find the numbers.

45. Divide 19 into two such parts that twice the smaller part exceeds the greater by two.

46. Three times the excess of a certain number over 6 is equal to the number plus 18. Find the number.

47. Thirty-one times a number exceeds 80 by as much as nine times the number is less than 80. Find the number.

48. Find the number whose double diminished by 3 exceeds 80 by as much as the number itself is less than 100.

49. Divide 19 into two parts such that the greater part exceeds twice the smaller part by 1 less than twice the smaller part.

50. A man is now twice as old as his son; 20 years ago he was four times as old as his son. Find the age of each.

51. A man is four times as old as his son; in 20 years he will be only twice as old. Find the age of each.

52. A man was four times as old as his son 7 years ago, and will be only twice as old as his son 7 years hence. Find the age of each.

53. A man has 8 hours for an excursion. How far can he ride into the country in a carriage that goes at the rate of 9 miles an hour so as to return in time, walking back at the rate of 3 miles an hour?

54. A man was hired for 26 days. Every day he worked he was to receive $3, and every day he was idle he was to pay $1 for his board. At the end of the time he received $62. How many days did he work?

55. A, walking 4 miles an hour, starts two hours after B, who walks 3 miles an hour. How many miles must A walk to overtake B?

56. A river runs 1 mile an hour. A man swims a certain distance up the river in 3 hours, and the same distance down in 1 hour. Find his rate of swimming in still water.

57. A man bought 12 yards of velvet. If he had bought 1 yard less for the same money, each yard would have cost $1 more. What did the velvet cost a yard?

58. A and B have together $8; A and C, $10; B and C, $12. How much has each?

59. I have in mind a certain number. If this number is diminished by 8 and the remainder multiplied by 8, the result is the same as if the number were diminished by 6 and the remainder multiplied by 6. What is the number?

60. A man having only ten-cent pieces and five-cent pieces wished to give some children 15 cents each, but found that he had not money enough by 25 cents; he, therefore, gave them 10 cents each and had 30 cents left. How many children were there?

61. A sum of money was divided among A, B, and C in such a way that A received three times as much as B, and B twice as much as C. If A received $6 more than C, how much did each receive?

62. The sum of the ages of a man and his son is 80 years; and the father's age is 2 years more than twice the age of his son. What is the age of each?

63. Two casks contain equal quantities of vinegar. From one cask 37 gallons are drawn, and from the other 7 gallons are drawn. The quantity now remaining in one cask is 7 times that remaining in the other. How much did each cask contain at first?

64. A merchant has two kinds of tea; one worth 50 cents a pound, and the other 75 cents a pound. He makes a mixture from these of 100 pounds, worth 60 cents a pound. How many pounds of each kind does he take?

65. A had $7 and B had $5. B gave A a certain sum; then A had 3 times as much as B. How many dollars did B give A?

66. A boy bought 9 dozen oranges for $2.50. For a part he paid 25 cents a dozen, and for the remainder 30 cents a dozen. How many dozen of each kind did he buy?

Note. In the following examples express in cents all money values.

67. How can \$2.25 be paid in quarters and ten-cent pieces so as to pay twice as many ten-cent pieces as quarters?

68. I have \$1.80 in ten-cent pieces and five-cent pieces, and have four times as many five-cent pieces as ten-cent pieces. How many have I of each?

69. I have \$6 in silver half-dollars and quarters, and I have 20 coins in all. How many have I of each?

70. I have five times as many half-dollars as quarters, and the half-dollars and quarters amount to \$11. How many have I of each?

71. A man has \$65 in ten-dollar bills and one-dollar bills. He has three times as many one-dollar bills as ten-dollar bills. How many bills has he of each kind?

72. A sum of money is divided among three persons, A, B, and C, in such a way that A and B together have \$6, A and C \$6.50, and B and C \$7.50. How much has each?

73. A purse contains 27 coins which amount to \$11.25. There is a certain number of silver dollars, and three times as many half-dollars as dollars; the remaining coins are quarters. Find the number of each.

74. A man bought 10 yards of calico and 20 yards of cloth for \$30.60. The cloth cost as many quarters per yard as the calico cost cents per yard. Find the price of each per yard.

75. A man has a certain number of dollars, half-dollars, and quarters. The number of quarters is twice the number of half-dollars and four times the number of dollars. If he has \$15, how many coins of each kind has he?

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CHAPTER III.

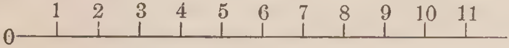
POSITIVE AND NEGATIVE NUMBERS.

61. Positive and Negative Quantities. If a person is engaged in trade, his capital will be *increased* by his *gains*, and *diminished* by his *losses.*

Increase in temperature is measured by the number of degrees the mercury *rises* in a thermometer, and *decrease* in temperature by the number of degrees the mercury *falls.*

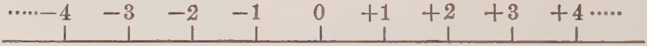
In considering any quantity whatever, a quantity that *increases* the quantity under consideration is called a *positive quantity;* and a quantity that *decreases* the quantity under consideration is called a *negative quantity.*

62. The Natural Series of Numbers. If from a given point, marked 0, we draw a straight line to the right, and beginning from the *zero* point lay off units of length on this line, the successive repetitions of the unit will be expressed by the *natural series of numbers,* 1, 2, 3, 4, etc. Thus:



In this series if we wish to *add* 2 to 5, we begin at 5, count 2 units *forwards,* and arrive at 7. If we wish to *subtract* 2 from 5, we begin at 5, count 2 units *backwards,* and arrive at 3. If we wish to subtract 5 from 5, we count 5 units backwards from 5, and arrive at 0. If we wish to subtract 5 from 2, we cannot do it, because when we have counted backwards from 2 as far as 0, *the natural series of numbers comes to an end.*

63. Positive and Negative Numbers. In order to subtract a greater number from a smaller it is necessary to *assume* a new series of numbers, beginning at zero and extending to the left of zero. The series to the left of zero must proceed from zero by *the repetitions of the unit,* precisely like the natural series to the right of zero; and the *opposition* between the right-hand series and the left-hand series must be clearly marked. This opposition is indicated by calling every number in the right-hand series a **positive number,** and prefixing to it, when written, the sign +; and by calling every number in the left-hand series a **negative number,** and prefixing to it the sign −. The two series of numbers may be called the **algebraic series of numbers,** and written thus:



If, in this double series of numbers, we wish to subtract 4 from 2, we begin at 2 in the positive series, count 4 units in the *negative direction* (to the left), and arrive at − 2 in the negative series; that is, $2 - 4 = -2$.

The result obtained by subtracting a greater number from a less, when both are positive, is *always a negative number.*

In general, if a and b represent any two numbers of the positive series, the expression $a - b$ will be a positive number when a is greater than b; will be zero when a is equal to b; will be a negative number when a is less than b.

In counting from left to right in the algebraic series, numbers *increase* in magnitude; in counting from right to left, numbers *decrease* in magnitude. Thus, − 3, − 1, 0, + 2, + 4, are arranged in *ascending* order of magnitude.

64. The Absolute Value of a Number. The absolute value of a number is its value independent of its sign.

65. Every algebraic number, as $+4$ or -4, consists of a *sign* $+$ or $-$ and the *absolute value* of the number. The sign shows whether the number belongs to the positive or negative series of numbers; the absolute value shows the place the number has in the positive or negative series.

When no sign stands before a number, the sign $+$ is always understood. But *the sign* $-$ *is never omitted.*

66. Two algebraic numbers that have, one the sign $+$, and the other the sign $-$, are said to have **unlike signs.**

Two algebraic numbers that have the same absolute values, but unlike signs, cancel each other when combined. Thus, $+4-4=0$; $+a-a=0$.

67. Double Meanings of the Signs + and −. The use of the signs $+$ and $-$ to indicate addition and subtraction must be carefully distinguished from the use of the signs $+$ and $-$ to indicate in which series, the positive or the negative, a given number belongs. In the first sense they are signs of *operations,* and are common to Arithmetic and Algebra; in the second sense they are signs of *opposition,* and are employed in Algebra alone.

NOTE. In Arithmetic, if the things counted are *whole units*, the numbers that count them are called **whole numbers, integral numbers,** or **integers,** the adjective being transferred from the things counted to the numbers that count them. But if the things counted are only *parts of units*, the numbers that count them are called **fractional numbers,** or simply **fractions,** the adjective being transferred from the things counted to the numbers that count them.

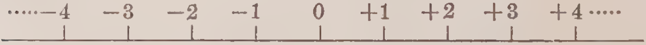
Likewise in Algebra, if the units counted are *negative*, the numbers that count them are called **negative numbers,** the adjective that defines the nature of the units counted being transferred to the numbers that count them.

68. Addition of Algebraic Numbers. An algebraic number is often enclosed in a parenthesis, in order that the signs $+$

and $-$, which are used to distinguish positive and negative numbers, may not be confounded with the $+$ and $-$ signs that denote the operations of addition and subtraction.

Thus, $+4+(-3)$ expresses the sum, and $+4-(-3)$ expresses the difference, of the numbers $+4$ and -3.

69. In order to add two algebraic numbers, we begin at the place in the series which the first number occupies, and count, *in the direction indicated by the sign of the second number*, as many units as there are in the absolute value of the second number.



Thus, the sum of $+2+(+3)$ is found by counting from $+2$ three units in *the positive direction*; that is, *to the right* and is, therefore, $+5$.

The sum of $+2+(-3)$ is found by counting from $+2$ three units in *the negative direction*; that is, *to the left*, and is, therefore, -1.

The sum of $-2+(+3)$ is found by counting from -2 three units in the positive direction, and is, therefore, $+1$.

The sum of $-2+(-3)$ is found by counting from -2 three units in the negative direction, and is, therefore, -5.

70. If a and b represent any two numbers, we have

$$+a+(+b)=a+b. \qquad -a+(+b)=-a+b.$$
$$+a+(-b)=a-b. \qquad -a+(-b)=-a-b.$$

Therefore, from these four cases, we have the following

Rule for Adding Two Algebraic Numbers:

1. *If the numbers have like signs, find the sum of their absolute values, and prefix the common sign to the result.*

2. *If the numbers have unlike signs, find the difference of their absolute values, and prefix the sign of the greater number to the result.*

71. The result is called **the algebraic sum** in distinction from the *arithmetical sum ;* that is, the sum of the absolute values of the numbers.

Note. If there are more than two numbers to be added, add two of the numbers, and then this sum to a third number, and so on ; or find the sum of the positive numbers and the sum of the negative numbers, then the difference between the absolute values of these two sums, and prefix the sign of the greater sum to the result.

Exercise 9.

Perform mentally the indicated additions :

1.	2.	3.	4.	5.	6.
+8	−8	−7	−11	+11	−11
−7	+7	−7	+ 4	− 4	− 4

7.	8.	9.	10.	11.	12.
+5	−8	−9	−20	+87	−37
−7	−7	−7	+24	−36	+40
−9	+9	−6	+36	−42	−20

13.	14.	15.	16.	17.	18.
+15	−15	−21	−20	−18	+17
+12	−12	+12	−30	+32	−27
−20	+30	−13	+40	−12	−19
					-39

19.	20.	21.	22.	23.	24.
31	−31	81	19	−15	90
−17	−17	−71	− 7	−70	−30
−15	−15	31	13	−40	−20
21	21	−40	−28	−90	10

Addition of Similar Monomials.

1. Find the sum of $3a$, $2a$, a, $5a$, $7a$.

The sum of the coefficients is $3 + 2 + 1 + 5 + 7 = 18$.

Hence, the sum of the monomials is $18a$.

2. Find the sum of $-5c$, $-c$, $-3c$, $-4c$, $-2c$.

The sum of the coefficients is $-5 - 1 - 3 - 4 - 2 = -15$.

Hence, the sum of the monomials is $-15c$.

3. Find the sum of $8x$, $-9x$, $-x$, $3x$, $4x$, $-12x$, x.

The sum of the positive coefficients is $8 + 3 + 4 + 1 = 16$.
The sum of the negative coefficients is $-9 - 1 - 12 = -22$.
The difference between 16 and 22 is 6, and the sign of the greater is negative. Hence, the sum is $-6x$. Therefore,

72. To Find the Sum of Similar Monomials,

Find the algebraic sum of the coefficients, and annex to this sum the letters common to the terms.

Exercise 10.

Perform mentally the indicated additions:

1.	2.	3.	4.	5.	6.
$7a$	$7xy$	$3yz$	$2ab$	$13c$	$3xz$
$2a$	$2xy$	$-9yz$	$-9ab$	$12c$	$7xz$
$-5a$	$-3xy$	$-2yz$	$-7ab$	$-24c$	$-9xz$
$-2a$	$4xy$	$7yz$	$8ab$	$15c$	$-xz$

7.	8.	9.	10.	11.	12.
$12z^4$	$4m^2$	$11abc$	$27y^4z^4$	$17x^3$	$18y^5$
$-7z^4$	$-9m^2$	$3abc$	$-41y^4z^4$	$-31x^3$	$27y^5$
$-8z^4$	$-7m^2$	$-7abc$	$-2y^4z^4$	$-47x^3$	$-43y^5$
$-z^4$	$5m^2$	$-9abc$	$-y^4z^4$	$61x^3$	$-21y^5$

EXERCISE 11.

Find the algebraic sum of:

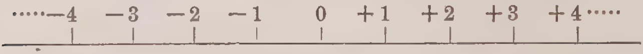
1. $7a, 2a, -3a, -5a.$
2. $7xy, 2xy, -4xy, -5xy.$
3. $4a^2b, -3a^2b, -5a^2b.$
4. $3xy, 4xy, 7xy, -3xy.$
5. $16b, -11b, -2b, 3b.$
6. $13c, 12c, -24c, 2c.$
7. $-3xz, 7xz, -2xz, -xz.$
8. $2ac, 5ac, -9ac, 3ac.$
9. $3yz, -9yz, 20yz, 7yz.$
10. $2ab, -10ab, -7ab, 3ab.$
11. $14x^2, 9x^2, -8x^2, -11x^2.$
12. $17y^3, -8y^3, 5y^3, -y^3.$
13. $12z^4, -7z^4, -8z^4, -9z^4.$
14. $21y^2, -17y^2, -3y^2, -4y^2.$
15. $c^2, -2c^2, -15c^2, -18c^2.$
16. $4m^2, -11m^2, -7m^2, 5m^2.$

Express in one term each of the following:

17. $9x^2 - 7x^2 + 4x^2 - 3x^2 + 3x^2 - 5x^2.$
18. $3a^2 - 18a^2 + a^2 - 5a^2 + 6a^2 - 10a^2.$
19. $5a^3x + 7a^3x - 9a^3x - 29a^3x + 4a^3x.$
20. $-5a^2b^2 + 7a^2b^2 + 11a^2b^2 - 4a^2b^2 - 9a^2b^2.$
21. $-21ax^3 + 20ax^3 - 6ax^3 + 5ax^3 - 13ax^3.$
22. $-11abcx + 3abcx - 7abcx + 29abcx + abcx.$
23. $-3y^4z^4 - 27y^4z^4 - 2y^4z^4 + 41y^4z^4 + y^4z^4.$
24. $-4x^5y^5 + 18x^5y^5 + 27x^5y^5 - 43x^5y^5 - x^5y^5.$
25. $-31abz^3 + 17abz^3 - 47abz^3 + 61abz^3 + abz^3.$
26. $\frac{1}{3}a + \frac{1}{2}a - \frac{7}{12}a + \frac{5}{6}a - \frac{3}{4}a + a.$ -1a 6
27. $\frac{5}{3}a^2 - 5a^2 + \frac{5}{6}a^2 + 7a^2 - \frac{3}{4}a^2 + \frac{1}{2}a^2.$
28. $xy - \frac{1}{2}xy + \frac{3}{4}xy - \frac{2}{3}xy + \frac{1}{6}xy - \frac{11}{12}xy.$
29. $\frac{2}{3}z - \frac{5}{6}z - \frac{3}{4}z + z + \frac{19}{12}z - \frac{5}{24}z.$
30. $\frac{11}{3}ayz + \frac{11}{4}ayz + \frac{11}{6}ayz + \frac{11}{2}ayz - 7ayz.$
31. $14a - 7a - 3a - 15a + 12a.$
32. $4a^2c - 10a^2c + 6a^2c - 9a^2c + a^2c.$
33. $3x^2y - 4x^2y + 2x^2y - x^2y + 5x^2y.$

Subtraction of Algebraic Numbers.

73. In order to subtract one algebraic number from another, we begin at the place in the series which the minuend occupies, and count, *in the direction opposite to that indicated by the sign of the subtrahend*, as many units as there are in the absolute value of the subtrahend.



Thus, the result of subtracting $+1$ from $+3$ is found by counting from $+3$ one unit in the *negative direction*; that is, in the direction *opposite to that indicated by the sign* $+$ *before* 1, and is, therefore, $+2$.

The result of subtracting -1 from $+3$ is found by counting from $+3$ one unit in the *positive direction*, and is, therefore, $+4$.

The result of subtracting $+1$ from -3 is found by counting from -3 one unit in the *negative direction*, and is, therefore, -4.

The result of subtracting -1 from -3 is found by counting from -3 one unit in the *positive direction*, and is, therefore, -2.

If a and b represent any two numbers, we have

$$+a-(+b)=a-b. \qquad -a-(+b)=-a-b.$$
$$+a-(-b)=a+b. \qquad -a-(-b)=-a+b.$$

74. From these four cases we see that *subtracting a positive number* is equivalent to *adding an equal negative number*; and that *subtracting a negative number* is equivalent to *adding an equal positive number.* Therefore,

75. To Subtract One Algebraic Number from Another,

Change the sign of the subtrahend, and add the result to the minuend.

EXERCISE 12.

Perform mentally the indicated subtractions:

1.	2.	3.	4.	5.	6.
11	-11	-11	11	3	-3
3	-3	3	-3	11	-11

7.	8.	9.	10.	11.	12.
3	3	-3	3	-3	3
8	-8	-8	-9	-9	9

13.	14.	15.	16.	17.	18.
-8	-7	8	7	7	6
7	8	-7	8	-3	-4

Subtraction of Similar Monomials.

1. From $15\,m^2x^2$ take $-7\,m^2x^2$.

$$\begin{aligned}15\,m^2x^2-(-7\,m^2x^2)&=15\,m^2x^2+7\,m^2x^2\\&=22\,m^2x^2.\end{aligned}$$ Hence,

76. To Subtract a Monomial from a Similar Monomial,

Change the sign of the coefficient of the subtrahend; then add the coefficients, and annex the common letters to the result.

EXERCISE 13.

Perform mentally the indicated subtractions:

1.	2.	3.	4.	5.
$16\,b$	$3\,yz$	$-10\,ab$	$-6\,a^2$	$-7\,m^2$
$-11\,b$	$-9\,yz$	$-7\,ab$	$-10\,a^2$	$3\,m^2$

6.	7.	8.	9.	10.
$29\,c$	$-41\,c$	$-29\,c$	$3\,abc$	$-ac^2x$
$-41\,c$	$29\,c$	$-41\,c$	$-abc$	$3\,ac^2x$

If $a = 4$, $b = -2$, $c = -3$, find the value of:

11. $a + (-b) + c$.
12. $-a - (-b) + c$.
13. $a - b + (-c)$.
14. $-(-a) + (-b) - (-c)$.
15. $+(-a) - (-b) - (-c)$.
16. $-(-a) - (-b) + (-c)$.

Multiplication of Algebraic Numbers.

77. **Multiplication** is generally defined in Arithmetic as the process of finding the result when one number (the multiplicand) is taken as many times as there are units in another number (the multiplier). This definition fails when the *multiplier is a fraction.* In multiplying by a fraction, we divide the multiplicand into as many **equal parts** as there are units in the denominator, and take as many of these parts as there are units in the numerator.

If, for example, we multiply 6 by $\frac{2}{3}$, we divide 6 into *three equal parts* and take *two* of these parts, obtaining 4 for the product. The multiplier, $\frac{2}{3}$, is $\frac{2}{3}$ of 1, and the product, 4, is $\frac{2}{3}$ of 6; that is, *the product is obtained from the multiplicand precisely as the multiplier is obtained from 1.*

78. Multiplication may be defined, therefore,

As the process of obtaining the product from the multiplicand as the multiplier is obtained from unity.

79. Every *extension of the meaning of a term* must be consistent with the sense previously attached to the term, and with the general laws of numbers.

This extension of the *meaning of multiplication* is consistent with the sense attached to multiplication when the multiplier is a positive whole number.

Thus, $5 \times 7 = 35$,
the multiplier, 5, $= 1 + 1 + 1 + 1 + 1$,
and the product, 35, $= 7 + 7 + 7 + 7 + 7$.

Law of Signs in Multiplication.

By the definition of multiplication (§ 78),

since $$+3 = +1 + 1 + 1,$$
$$3 \times (+8) = +8 + 8 + 8 = +24,$$
and $$3 \times (-8) = -8 + (-8) + (-8) = -24.$$

Again, since $$-3 = -1 - 1 - 1,$$
$$(-3) \times 8 = -8 - 8 - 8 = -24,$$
and $$(-3) \times (-8) = -(-8) - (-8) - (-8)$$
$$= +8 + 8 + 8 = +24.$$

The *minus sign before the multiplier*, 3, signifies that the repetitions of the multiplicand are to be *subtracted*.

If a and b stand for any two numbers, we have

$$(+a) \times (+b) = +ab,$$
$$(+a) \times (-b) = -ab,$$
$$(-a) \times (+b) = -ab,$$
$$(-a) \times (-b) = +ab.$$

That is, if two numbers have **like** signs, the product has the **plus** sign; if **unlike** signs, the product has the **minus** sign.

80. **The Law of Signs in Multiplication** is, therefore,

Like signs give +, and unlike signs give −.

The Index Law in Multiplication.

Since $$a^2 = aa, \text{ and } a^3 = aaa,$$
$$a^2 \times a^3 = aa \times aaa = a^5 = a^{2+3};$$
$$a^4 \times a = aaaa \times a = a^5 = a^{4+1}.$$

If then m and n are *positive integers*,

$$a^m \times a^n = a^{m+n}.$$

In like manner, $a^m \times a^n \times a^p = a^{m+n+p}$.

81. **The Index Law in Multiplication** is, therefore,

The exponent of a letter in the product is equal to the sum of the exponents of the letter in the factors of the product.

Multiplication of Monomials.

1. Find the product of $6a^2b^3$ and $7ab^2c^3$.

Since the order of the factors is immaterial, (§ 42)

$$6a^2b^3 \times 7ab^2c^3 = 6 \times 7 \times a^2 \times a \times b^3 \times b^2 \times c^3$$
$$= 42a^3b^5c^3.$$

2. Find the product of $-3ab$ and $7ab^3$.

$$-3ab \times 7ab^3 = -3 \times 7 \times a \times a \times b \times b^3$$
$$= -21a^2b^4.$$

3. Find the product of x^n and x^3, and of x^n and x^n.

$$x^n \times x^3 = x^{n+3}.$$
$$x^n \times x^n = x^{n+n} = x^{2n}.$$ Therefore,

82. To Find the Product of Two Monomials,

Find the product of the numerical coefficients; and to this product annex the letters, giving to each letter an exponent equal to the sum of its exponents in the factors.

83. A product of three or more factors is called **the continued product** of the factors.

1. Find the continued product of $(-a)\times(-b)\times(-c)$.

By the law of signs, § 80, we have

$$(-a) \times (-b) = ab,$$
and
$$(ab) \times (-c) = -abc.$$

2. Find the continued product of

$$(-a) \times (-b) \times (-c) \times (-d).$$

By the law of signs,
$$(-a) \times (-b) = ab,$$
$$(ab) \times (-c) = -abc,$$
$$(-abc) \times (-d) = abcd.$$

84. From Examples 1 and 2 (§ 83), we see that an **odd number of negative factors** gives a **negative product**; and an **even number of negative factors** gives a **positive product.**

EXERCISE 14.

NOTE. The beginner should first write the *sign* of the product; then the product of the numerical coefficients after the sign; and, lastly, the letters in alphabetical order, giving to each letter the proper exponent.

Find mentally the product of:

1.	2.	3.	4.	5.	6.
$3a$	$-3a$	$3a$	$-3a$	$9a^2$	$-9a^2$
$2a$	$-2a$	$-2a$	$2a$	$6a$	$6a^2$

7.	8.	9.	10.	11.	12.
$5x^2$	$-7a^4$	$8a^3$	$-7m^2$	x^2y^2	$-a^2b^3c^4$
$7x^5$	$-a$	$-7a^3$	$8m^4$	$-5x^2y^2$	a^2bc

13.	14.	15.	16.	17.	18.
$-3a^2x$	$5ac^3$	$7x^5$	$-9b^2$	$-a^3b$	$9a^m$
$-3ax^2$	$6a^2c$	$-3x^7$	$-6a^2$	$-ab^4$	$6a^n$

19.	20.	21.	22.	23.	24.
a^{m+1}	$-x^n$	z^{n+3}	a^n	y^n	x^2
a^{m-1}	$2x^{m-n}$	z^{n-2}	a	y^{n+2}	x^{n-1}

25. $-2x^2y$, x^2y^3, and $-15abxy$.
26. $-8a^2$, $-2b^2$, $-3ab$, and a^2b^2.
27. ab, $-ac$, $-bc$, and $-3abc$.
28. $-xyz^2$, x^2yz, $-xy^2z$, and $3xyz$.
29. $4a^3$, $-10a^2b$, $25ab^2$, and $-ab^4$.
30. b^2, $6ab^2$, $-4a^2b$, and $-2ab^2$.
31. a^{3n}, a^{2n-1}, a^{2n+1}, a^{1-2n}, and a.
32. x^{m+1}, x^{m-2}, x^m, x^{m+n}, and x^{m-n}.

Division of Algebraic Numbers.

85. **Division** is the operation of finding one of two factors, when their product and the other factor are given.

86. With reference to this operation the product is called the **dividend**, the given factor the **divisor**, and the required factor the **quotient**.

Law of Signs in Division.

Since $(+a) \times (+b) = +ab$, $\therefore +ab \div (+a) = +b$.
Since $(+a) \times (-b) = -ab$, $\therefore -ab \div (+a) = -b$.
Since $(-a) \times (+b) = -ab$, $\therefore -ab \div (-a) = +b$.
Since $(-a) \times (-b) = +ab$, $\therefore +ab \div (-a) = -b$.

That is, if the dividend and divisor have **like** signs, the quotient has the **plus** sign; and if they have **unlike** signs, the quotient has the **minus** sign.

87. **The Law of Signs in Division** is, therefore,

Like signs give $+$, *and unlike signs give* $-$.

The Index Law in Division.

The quotient contains the factors of the dividend that are not found in the divisor.

1. Divide a^5 by a^2.   2. Divide a^4 by a.

$$1.\quad \frac{a^5}{a^2} = \frac{aaaaa}{aa} = aaa = a^3 = a^{5-2}.$$

$$2.\quad \frac{a^4}{a} = \frac{aaaa}{a} = aaa = a^3 = a^{4-1}.$$

If m and n are positive integers, and m is greater than n,

$$a^m \div a^n = a^{m-n}.$$

88. The Index Law in Division is, therefore,

The exponent of a letter in the quotient is equal to the exponent of the letter in the dividend minus the exponent of the letter in the divisor.

Division of Monomials.

1. Divide $24\,a^7$ by $8\,a^5$.

$$\frac{24\,a^7}{8\,a^5} = 3\,a^{7-5} = 3\,a^2.$$

We obtain the factor 3 of the quotient by dividing 24 by 8; and the factor a^2 of the quotient, by writing a with an exponent equal to the exponent of a in the dividend minus the exponent of a in the divisor.

2. Divide $20\,a^5b^6$ by $-4\,ab^5$.

$$\frac{20\,a^5b^6}{-4\,ab^5} = -5\,a^{5-1}b^{6-5} = -5\,a^4b.$$

3. Divide $-30\,a^2b^3c^4$ by $-20\,abc^3$.

$$\frac{-30\,a^2b^3c^4}{-20\,abc^3} = \tfrac{3}{2}\,ab^2c.$$

4. Divide $-57\,a^{x-1}$ by $-19\,a^{x-3}$.

$$\frac{-57\,a^{x-1}}{-19\,a^{x-3}} = 3\,a^{x-1-(x-3)} = 3\,a^{x-1-x+3} = 3\,a^2.$$

5. Divide $77\,a^{2m}b^nc^x$ by $11\,a^mb^nc^3$.

$$\frac{77\,a^{2m}b^nc^x}{11\,a^mb^nc^3} = 7\,a^{2m-m}\,b^{n-n}\,c^{x-3} = 7\,a^mb^0c^{x-3}.$$

NOTE. Since by division $\frac{b^n}{b^n} = 1$; and by the index law $\frac{b^n}{b^n} = b^0$, it follows that $b^0 = 1$. Hence, any *letter* which by the rule would appear in the quotient *with zero for an exponent*, may be omitted without affecting the quotient.

89. To Find the Quotient of Two Monomials, therefore,

Divide the numerical coefficient of the dividend by the numerical coefficient of the divisor; and to the result annex the letters, giving to each letter an exponent equal to its exponent in the dividend minus its exponent in the divisor.

Exercise 15.

Perform mentally the indicated divisions:

1. $\frac{12\,ab^2}{6\,ab}$
2. $\frac{16\,a^2c^3}{8\,ac}$
3. $\frac{20\,xy^3}{5\,xy}$
4. $\frac{14\,x^3y^3z^3}{7\,xy^2z^3}$
5. $\frac{54\,a^4x^3c^4}{6\,a^3x^2c}$
6. $\frac{-20\,c^3y^3}{5\,c^2y}$
7. $\frac{-27\,a^4x^2}{-9\,a^3}$
8. $\frac{-56\,a^4x^4}{-8\,a^3x}$
9. $\frac{63\,y^2z^3}{-7\,yz^2}$
10. $\frac{72\,x^3yz^2}{12\,xyz}$
11. $\frac{-9\,a^2b^2c^3}{-ab^2c}$
12. $\frac{-27\,ab^2x^3}{3\,abx^2}$
13. $\frac{-3\,x^4y^4z^4}{x^2y^2z^3}$
14. $\frac{-2\,c^3y^3}{-c^2y^3}$
15. $\frac{-18\,x^n}{-4\,x^{n-2}}$
16. $\frac{56\,a^2x^3y^4}{-7\,x^3y^2}$
17. $\frac{-3\,ab^4x^2}{-ab^2x^2}$
18. $\frac{-2\,a^2b^2z^4}{abz^2}$
19. $\frac{4\,x^3y^3}{-2\,x^2y^3}$
20. $\frac{6\,a^4x^2}{-2\,ax^2}$
21. $\frac{-a^5x^9}{-a^3x^2}$
22. $\frac{-51\,x^3y^3}{17\,xy^2}$
23. $\frac{28\,c^3d^4}{-7\,c^2d^2}$
24. $\frac{16\,ab^2x^3}{-4\,ab^2x}$
25. $\frac{12\,x^3y^3z^4}{-3\,xy^2z^3}$
26. $\frac{7\,a^2bc^n}{7\,abc^{n-1}}$
27. $\frac{3\,a^{n+1}}{-a^{n-1}}$
28. $\frac{12\,x^ny^n}{6\,x^2y}$
29. $\frac{54\,x^{n-2}y^{n-1}}{-9\,x^2y^2}$

$x^{n-4}\,y^{n}$

30. $\frac{x^3y^8}{x^{n-2}y^{n-1}}$
31. $\frac{-6\,c^2y^2}{-2\,cy}$
32. $\frac{4\,a^4b^4}{2\,a^2b^2}$
33. $\frac{-3\,a^5b^5}{-3\,a^4b}$
34. $\frac{2\,y^3z^2}{2\,y^3z}$
35. $\frac{-y^7z^8}{-y^5z^3}$
36. $\frac{-8\,c^9y^3}{-4\,c^6y^2}$
37. $\frac{10\,a^{10}x^5}{5\,a^8x^5}$
38. $\frac{12\,a^{12}b^8}{6\,a^9b^8}$
39. $\frac{3\,x^9y^{10}}{-x^8y^9}$
40. $\frac{-x^{10}z^{12}}{-x^9z^{10}}$

CHAPTER IV.

ADDITION AND SUBTRACTION.

Integral Compound Expressions.

90. If an algebraic expression contains no *letter in the denominator of any of its terms,* it is called an **integral expression.** Thus, $x^3 + 7\,cx^2 - c^3 - 5\,c^2x$, $\frac{1}{2}\,ax - \frac{1}{3}\,bcy$, are integral expressions.

An integral expression may have for some values of the letters a fractional value, and a fractional expression an integral value. If, for instance, a stands for $\frac{3}{4}$ and b for $\frac{1}{4}$, the integral expression $2\,a - 5\,b$ stands for $\frac{6}{4} - \frac{5}{4} = \frac{1}{4}$; and the fractional expression $\frac{5\,a}{3\,b}$ stands for $\frac{15}{4} \div \frac{3}{4} = 5$. Integral and fractional expressions, therefore, are so named on account of the *form of the expressions*, and with no reference whatever to the numerical value of the expressions when definite numbers are put in place of the letters.

Addition of Integral Compound Expressions.

91. The addition of two compound algebraic expressions can be represented by connecting the second expression with the first by the sign +. If there are no like terms in the two expressions, the operation is *algebraically complete* when the two expressions are thus connected (§ 11, Note).

If, for example, it is required to add $m + n - p$ to $a + b + c$, the result will be $a + b + c + (m + n - p)$; or, removing the parenthesis (§ 39), $a + b + c + m + n - p$.

92. If, however, there are like terms in the expressions, every set of like terms can be replaced by a single term with a coefficient equal to the algebraic sum of the coefficients of the like terms.

1. Add $5a^2 + 4a + 3$ to $2a^2 - 3a - 4$.

$$\begin{aligned} &2a^2 - 3a - 4 + (5a^2 + 4a + 3) \\ &\qquad = 2a^2 - 3a - 4 + 5a^2 + 4a + 3 && \text{(§ 39)} \\ &\qquad = 2a^2 + 5a^2 - 3a + 4a - 4 + 3 && \text{(§ 38)} \\ &\qquad = 7a^2 + a - 1. \end{aligned}$$

This process is more conveniently represented by arranging the terms in columns, so that *like terms* shall stand in the *same column*, as follows:

$$\begin{array}{rrr} 2a^2 & -\ 3a & -\ 4 \\ 5a^2 & +\ 4a & +\ 3 \\ \hline 7a^2 & +\ \ a & -\ 1 \end{array}$$

The coefficient of a^2 in the result will be $5 + 2$, or 7; the coefficient of a will be $-3 + 4$, or 1; the last term will be $-4 + 3$, or -1.

2. Add $2a^3 - 3a^2b + 4ab^2 + b^3$; $a^3 + 4a^2b - 7ab^2 - 2b^3$; $3a^3 + a^2b - 3ab^2 - 4b^3$; and $2a^3 + 2a^2b + 6ab^2 - 3b^3$.

$$\begin{array}{rrrr} 2a^3 & -\ 3a^2b & +\ 4ab^2 & +\ \ b^3 \\ a^3 & +\ 4a^2b & -\ 7ab^2 & -\ 2b^3 \\ 3a^3 & +\ \ a^2b & -\ 3ab^2 & -\ 4b^3 \\ 2a^3 & +\ 2a^2b & +\ 6ab^2 & -\ 3b^3 \\ \hline 8a^3 & +\ 4a^2b & & -\ 8b^3 \end{array}$$

The coefficient of a^3 in the result will be $2 + 1 + 3 + 2$, or $+8$; the coefficient of a^2b will be $-3 + 4 + 1 + 2$, or $+4$; the coefficient of ab^2 will be $4 - 7 - 3 + 6$, or 0, and, therefore, the term ab^2 will not appear in the result (§ 21); and the coefficient of b^3 will be $1 - 2 - 4 - 3$, or -8.

EXERCISE 16.

Add:

1. $a + b;\ a - b.$

2. $x^2 - x;\ x^3 - x^2.$

3. $5x^2 + 6x - 2;\ 3x^2 - 7x + 2.$

4. $3x^2 - 2xy + y^2;\ x^2 - 2xy + 3y^2.$

5. $ax^2 + bx - 4;\ 3ax^2 - 2bx + 4;\ -4ax^2 - 2bx + 5.$

6. $5x + 3y + z;\ 3x + 2y + 3z;\ x - 3y - 5z.$

7. $-3ab - 2ax^2 + 3a^2x + x^3;\ -4ab - 6a^2x + 5ax^2;$
$x^3 - ab + a^2x - ax^2;\ ax^2 + 8ab - 5a^2x.$

8. $a^4 - 2a^3 + 3a^2 - a + 7;\ 2a^4 - 3a^3 + 2a^2 - a + 6;$
$a^4 - 2a^3 + 2a^2 - 5.$

9. $3a^2 - ab + ac - 3b^2 + 4bc - c^2;\ -4bc + 5c^2 + 2ab;$
$5a^2 - ab - ac + 5bc;\ -4a^2 + b^2 - 5bc + 2c^2.$

10. $x^4 - 3x^3 + 2x^2 - 4x + 7;\ 3x^4 + 2x^3 + x^2 - 5x - 6;$
$4x^4 + 3x^3 - 3x^2 + 9x - 2;\ 2x^4 - x^3 + x^2 - x + 1.$

11. $7y^3 - 3xy^2 - 4x^2y + x^3;\ -5x^3 - 11xy^2 - 12xz^2 - y^3;$
$x^2y - xz^2 - y^2 - 5xy^2;\ -4xz^2 + y^2 - z^3 + 6xy^2 + 10x^3.$

12. $a^4 - 2a^3 + 3a^2 - 3a - 2;\ a^3 + a^2 + a - 3a^4 + 3;$
$4a^4 + 5a^3;\ 2a^2 + 3a - 2;\ -a^2 - 2a - 3.$

13. $x^3 + 2xy^2 - x^2y - y^3;\ 2x^3 - 3x^2y - 4xy^2 - 7y^3;$
$x^3 - 8xy^2 - 7y^3.$

14. $c^4 - 3c^3 + 2c^2 - 4c + 7;\ 2c^4 + 3c^3 + 2c^2 + 5c + 6;$
$c^3 - 4c^4 - 4c^2 - 5.$

15. $3x^2 - xy + xz - 3y^2 - z^2;\ -5x^2 - xy - xz + 5yz;$
$y^2 + 3yz + 3z^2;\ 6x^2 - 6y - 6z + 4xz;\ 4yz - 5xz.$

16. $m^5 - 3m^4n - 6m^3n^2;\ m^3n^2 + m^2n^3 - 5m^4n - 3n^5;$
$2m^5 + 4m^2n^3 - 3mn^4 - n^5;\ -2m^2n^3 - 3mn^4 + n^5;$
$-m^5 + 2mn^4 + 2n^5 + 3m^4n.$

17. $6y^3 - 1 - 2x^2y;\ 5 + 2xy^2 - 4x^2y;\ -x^2y - 5 + 6xy^2;$
$2 + xy^2 - y^3;\ x^2y - 2xy^2 - 5y^3 + 1.$

Subtraction of Compound Expressions.

93. The subtraction of one expression from another, if none of the terms are alike, can be represented only by connecting the subtrahend with the minuend by means of the sign $-$.

If, for example, it is required to subtract $a+b+c$ from $m+n-p$, the result will be represented by

$$m+n-p-(a+b+c);$$

or, removing the parenthesis (§ 40),

$$m+n-p-a-b-c.$$

If, however, some of the terms in the two expressions are alike, we can replace like terms by a single term:

1. Subtract a^3-2a^2+2a-1 from $3a^3-2a^2+a-2$; the result may be expressed as follows:

$$3a^3-2a^2+a-2-(a^3-2a^2+2a-1);$$

or, removing the parenthesis (§ 40),

$$\begin{aligned} &3a^3-2a^2+a-2-a^3+2a^2-2a+1 \\ &\quad =3a^3-a^3-2a^2+2a^2+a-2a-2+1 \qquad (§\ 38) \\ &\quad =2a^3-a-1. \end{aligned}$$

This process is more easily performed by writing the subtrahend below the minuend, mentally changing the sign of each term in the subtrahend, and adding the two expressions. Thus, the above example may be written

$$\begin{array}{r} 3a^3-2a^2+a-2 \\ a^3-2a^2+2a-1 \\ \hline 2a^3-a-1 \end{array}$$

The coefficient of a^3 will be $3-1$, or 2; the coefficient of a^2 will be $-2+2$, or 0, and therefore the term containing a^2 will not appear in the result; the coefficient of a will be $1-2$, or -1; the last term will be $-2+1$, or -1.

2. Subtract $a^5 + 4a^3x^2 - 3a^2x^3 - 4ax^4$
from $a^3x^2 + 2a^2x^3 - 4ax^4$.

Here terms that are alike can be written in columns:

$$\begin{array}{rrrr}
 & a^3x^2 & +\,2a^2x^3 & -\,4ax^4 \\
a^5 & +\,4a^3x^2 & -\,3a^2x^3 & -\,4ax^4 \\
\hline
-a^5 & -\,3a^3x^2 & +\,5a^2x^3 &
\end{array}$$

There is no term a^5 in the minuend, hence the coefficient of a^5 in the result will be $0 - 1$, or -1; the coefficient of a^3x^2 will be $1 - 4$, or -3; the coefficient of a^2x^3 will be $2 + 3$, or $+5$; the coefficient of ax^4 will be $-4 + 4$, or 0, and ax^4 will not appear in the result.

Exercise 17.

1. From $8a - 4b - 2c$ take $2a - 3b - 3c$.
2. From $3a - 4b + 3c$ take $2a - 8b - c - d$.
3. From $7a^2 - 9x - 1$ take $5a^2 - 6x - 3$.
4. From $2x^2 - 2ax + a^2$ take $x^2 - ax - a^2$.
5. From $4a - 3b - 3c$ take $2a - 3b + 4c$.
6. From $5x^2 + 7x + 4$ take $3x^2 - 7x + 2$.
7. From $2ax + 3by + 5$ take $3ax - 3by - 5$.
8. From $4a^2 - 6ab + 2b^2$ take $3a^2 + ab + b^2$.
9. From $4a^2b + 7ab^2 + 9$ take $8 - 3ab^2$.
10. From $5a^2c + 6a^2b - 8a^3$ take $b^3 + 6a^2b - 5a^2c$.
11. From $a^2 - b^2$ take b^2.
12. From $a^2 - b^2$ take a^2.
13. From b^2 take $a^2 - b^2$.
14. From a^2 take $a^2 - b^2$.
15. From $x^4 + 3ax^3 - 2bx^2 + 3cx - 4d$
take $3x^4 + ax^3 - 4bx^2 + 6cx + d$.

If $A = 3a^2 - 2ab + 5b^2$, $C = 7a^2 - 8ab + 5b^2$,
$B = 9a^2 - 5ab + 3b^2$, $D = 11a^2 - 3ab - 4b^2$,

find the expression for:

16. $A + C + B - D$.
17. $A - C - B + D$.
18. $C - A - B + D$.
19. $A + C - B - D$.
20. $A - C + B + D$.
21. $A + C - B + D$.

Insertion of Parentheses.

We have the following equivalent expressions:

$$a + (b + c) = a + b + c, \quad \therefore a + b + c = a + (b + c);$$
$$a + (b - c) = a + b - c, \quad \therefore a + b - c = a + (b - c);$$
$$a - (b + c) = a - b - c, \quad \therefore a - b - c = a - (b + c);$$
$$a - (b - c) = a - b + c, \quad \therefore a - b + c = a - (b - c).$$

94. Hence, a parenthesis preceded by the sign + may not only be removed *without changing the sign of any term*, but may also be inserted, enclosing any number of terms, *without changing the sign of any term.*

And a parenthesis preceded by the sign − may not only be removed, *provided the sign of every term within the parenthesis is changed*, namely, + to − and − to +, but may also be inserted, enclosing any number of terms, *provided the sign of every term enclosed is changed.*

95. Expressions may occur having parentheses within parentheses. In such cases signs of aggregation of different shapes are used, and the beginner, when he meets with one branch of a parenthesis (, or bracket [, or brace {, must look carefully for the other branch, whatever may intervene; and all that is included between the two branches must be treated as the + or − sign before the sign of aggregation directs. It is best to remove each parenthesis in succession, *beginning with the innermost.* Thus,

1. $$\begin{aligned} & a - [b - (c - d) + e] \\ &= a - [b - c + d + e] \\ &= a - b + c - d - e. \end{aligned}$$

2. $$\begin{aligned} & a - \{b - [c - (d - e) + f]\} \\ &= a - \{b - [c - d + e + f]\} \\ &= a - \{b - c + d - e - f\} \\ &= a - b + c - d + e + f. \end{aligned}$$

EXERCISE 18.

Simplify the following by removing the parentheses and combining like terms:

1. $a - b - [a - (b - c) - c]$.
2. $m - [n - (p - m)]$.
3. $2x - \{y + [4z - (y + 2x)]\}$.
4. $3a - \{2b - [5c - (3a + b)]\}$.
5. $a - \{b + [c - (d - b) + a] - 2b\}$.
6. $3x - [9 - (2x + 7) + 3x]$.
7. $2x - [y - (x - 2y)]$.
8. $a - [2b + (3c - 2b) + a]$.
9. $(a - x + y) - (b - x - y) + (a + b - 2y)$.
10. $3a - [-4b + (4a - b) - (2a - 5b)]$.
11. $4c - [a - (2b - 3c) + c] + [a - (2b - 5c - a)]$.
12. $x + (y - z) - [(3x - 2y) + z] + [x - (y - 2z)]$.
13. $a - [2a + (a - 2a) + 2a] - 5a - \{6a - [(a + 2a) - a]\}$.
14. $2x - (3y + z) - \{b - (c - b) + c - [a - (c - b)]\}$.
15. $a - [b + c - a - (a + b) - c] + (2a - \overline{b + c})$.

NOTE. Remember that the sign $-$ which is written in the last problem before the first term b under the vinculum is really the sign of the vinculum, $-\overline{b + c}$ meaning the same as $-(b + c)$.

16. $10 - x - \{-x - [x - (x - \overline{5 - x})]\}$.
17. $2x - \{2x + (y - z) - 3z + [2x - (y - \overline{z - 2y}) - 3z] + 4y\}$.
18. $a - \{b - [-c + a - (a - b) - c]\} + [2a - (b - a)]$.
19. $a - \{b - [a - (c - b) + \overline{c - a} - (a - b - c) - a] + a\}$.
20. $5a - \{-3a - [3a - (2a - \overline{a - b}) - a] + a\}$.
21. $20 - a - \{7a - [8a - (9a - \overline{3 - 6a})]\}$.
22. $x - \{5y - [x - (3y - 2z) + z - (x - 2y - z)]\}$.

EXERCISE 19.

In each of the following expressions enclose the last three terms in a parenthesis preceded by the sign −, remembering that the sign of each term enclosed must be changed:

1. $2a + b + 3c - d + 3e - 5f$.
2. $x - a - y - (b - z - c.)$
3. $a + b - c + 4a - b + 1$.
4. $ax + by + cz + bx - cy + cz$.
5. $3a + 2b + 2c - 5d - 3e - 4f$.
6. $x - y + z - 5xy - 4xz + 3yz$.

Considering all the factors that precede x, y, and z, respectively, as the *coefficients* of these letters, we may collect in parentheses the coefficients of x, y, and z in the following expression:

$$\begin{aligned} ax - by + ay - az - cz + bx &= ax + bx + ay - by - az - cz \\ &= (a + b)x + (a - b)y - (a + c)z. \end{aligned}$$

In like manner, collect the coefficients of x, y, and z in the following expressions:

7. $ax + by + cz + bx - cy + az$.
8. $ax + 2ay + 4az - bx + 3y - 3bz - 2z$.
9. $ax - 2by - 5cz - 4bx + 3cy - 7az$.
10. $ax + 3ay + 2by - bz - 11cx + 2cy - cz$.
11. $4by - 3ax - 6cz + 2bx - 7cx - 5cy - x - y - z$.
12. $6az - 5by + 3cz - 2bz - 3ay + z - x + y$.
13. $z - by + 3az - 3cy + 2ax - 2mx - 5bz$.
14. $x + ay - az - acx + bcz - mny - y - z$.
15. $2ax - 6ay + 4bz - 4bx - 2cx - 3cy$.
16. $ax - bx + 2ay + 3y + 4az - 3bz - 2z$.
17. $ax - 2by + 5cz - 4bx - 3cy + az - 2cx - ay + 4bz$.
18. $12ax + 12ay + 4by - 12bz - 15cx + 6cy + 3cz$.

Exercise 20. — Review.

1. Add $4x^3 - 5a^2 - 5ax^2 + 6a^2x$; $6a^3 + 3x^3 + 4ax^2$; $19ax^2 - 11x^2 - 15a^2x$; $10x^3 + 7a^2x + 5a^3 - 18ax^2$.

2. Add $3ab + 3a + 6b - 4b^2$; $ab + 2a + 4b + 9b^2$; $7ab - 4a - 8b + 13a^2$; $6a + 12b - 2ab - 11a^2$.

Note. *Similar compound expressions* are added in precisely the same way as simple expressions, by finding the sum of their coefficients. Thus, $3(x-y) + 5(x-y) - 2(x-y) = 6(x-y)$.

3. Add $4(5-x)$; $6(5-x)$; $3(5-x)$; $-12(5-x)$; $2(5-x)$; $-9(5-x)$.

4. Add $(a+b)x^2 + (b+c)y^2 + (a+c)z^2$; $(b+c)x^2 + (a+c)y^2 + (a+b)z^2$; $(a+c)x^2 + (a+b)y^2 + (b+c)z^2$.

5. Add $(a+b)x + (b+c)y + (c+a)z$; $(b+c)z + (c+a)x - (a+b)y$; $(a+c)y + (a+b)z - (b+c)x$.

6. From $a^3 - x^2$ take $a^3 + 2ax + x^2$.

7. From $3a^2 + 2ax + x^2$ take $a^2 - ax - x^2$.

8. From $8x^2 - 3ax + 5$ take $5x^2 + 2ax + 5$.

9. From $a^3 + 3b^2c + ab^2 - abc$ take $ab^2 - abc + b^3$.

10. From $(a+b)x + (a+c)y$ take $(a-b)x - (a-c)y$.

11. Simplify $7a - \{3a - [4a - (5a - 2a)]\}$.

12. Simplify $3a - \{a + b - [a + b + c - (a + b + c + d)]\}$.

Bracket the coefficients, and arrange according to the descending powers of x:

13. $x^2 - ax - c^2x^2 - bx + bx^3 - cx^2 + a^2x^3 - x^3 - cx$.

14. $2ax - 3bx^2 - 7cx^3 - 2bx + 2cx^2 + 8ax^3 - 2cx - ax^2 - bx^3$.

If $a = 1$, $b = 3$, $c = 5$, and $d = 7$, find the value of:

15. $a^2 - (b^2 - c^2) - [b^2 - (c^2 - a^2)] + [c^2 - (b^2 - a^2)]$.

16. $a - 2b - \{3c - d - [3a - (5b - c - 8d)] - 2b\}$.

17. From $2d + 11a + 10b - 5c$ take $2c + 5a - 3b$; and show that the result is numerically correct when $a = 1$, $b = 3$, $c = 5$, $d = 7$.

18. If $a = 1$, $b = -3$, $c = -5$, $d = 0$, find the value of $a^2 + 2b^2 + 3c^2 + 4d^2$.

If $a = 3$, $b = 4$, $c = 9$, and $2s = a + b + c$, find the value of:

19. $s(s-a)(s-b)(s-c)$.

20. $s^2 + (s-a)^2 + (s-b)^2 + (s-c)^2$.

21. $s^2 - (s-a)(s-b) - (s-b)(s-c) - (s-c)(s-a)$.

22. If $x = a + 2b - 3c$, $y = b + 2c - 3a$, $z = c + 2a - 3b$, show that $x + y + z = 0$.

23. If $x = a - 2b + 3c$, $y = b - 2c + 3a$, $z = c - 2a + 3b$, show that $x + y + z = 2a + 2b + 2c$.

24. What must be added to $x^2 + 5y^2 + 3z^2$ in order that the sum may be $2y^2 - z^2$?

25. What must be added to $5a^3 - 7a^2b + 3ab^2$ in order that the sum may be $a^3 - 2a^2b - 2ab^2 + b^3$?

$$E = 5a^3 + 3a^2b - 2b^3,\quad F = 3a^3 - 7a^2b - b^3,$$
$$G = 2a^2b - a^3 - b^3,\quad H = a^2b - 2a^3 - 3b^3,$$

If the above, find the simplest expression for:

26. $E + \{F - G - H\}$.

27. $E - \{F + G - H\}$.

28. $E + \{F - G + H\}$.

29. $E - \{F - (G - H)\}$.

30. $E + G - (F - H)$.

31. $F - H - (E + G)$.

32. $H - E - (F - G)$.

33. $F - G - (E - H)$.

34. From $2x^2 - 2y^2 - z^2$ take $3y^2 + 2x^2 - z^2$, and from the remainder take $3z^2 - 2y^2 - x^2$.

35. Take the sum of $a^2c - 2a^3 + 2ac^2$ and $a^3 - ac^2 - a^2c$ from $a^3 - 2a^2c + 3ac^2$.

CHAPTER V.

MULTIPLICATION AND DIVISION.

Multiplication of Compound Expressions.

96. Degree of a Term. A term that has *one letter* is said to be of the *first degree;* a term that is the product of *two letters* is said to be of the *second degree;* and so on.

97. Degree of a Compound Expression. The degree of a compound expression is the degree of that term of the expression which is of the *highest degree.*

Thus, $a^2x^2 + bx + c$ is of the fourth degree, since a^2x^2 is of the fourth degree.

98. When all the terms of a compound expression are of the same degree, the expression is said to be **homogeneous.**

Thus, $x^3 + 3x^2y + 3xy^2 + y^3$ is a homogeneous expression, every term being of the third degree.

99. Dominant Letter. If there is one letter in an expression of more importance than the rest, it is called the **dominant letter**; and the degree of the expression is called by the degree of the *dominant letter.*

Thus, $a^2x^2 + bx + c$ is of the *second degree in x.*

100. Arrangement of a Compound Expression. A compound expression is said to be *arranged* according to the powers of some letter when the exponents of that letter descend or ascend, from left to right, in *the order of magnitude.*

Thus, $3ax^3 - 4bx^2 - 6ax + 8b$ is arranged according to the descending powers of x; and $8b - 6ax - 4bx^2 + 3ax^3$ is arranged according to the ascending powers of x.

Multiplication of Polynomials by Monomials.

We have $a(b+c)=ab+ac$; (§ 41)

and $a(b-c)=ab-ac$. Hence,

101. To Multiply a Polynomial by a Monomial,

Multiply each term of the polynomial by the monomial, and connect the partial products with their proper signs.

Find the product of $ab+ac-bc$ and abc.

$$\begin{array}{r} ab+ac-bc \\ abc \\ \hline a^2b^2c+a^2bc^2-ab^2c^2 \end{array}$$

NOTE. We multiply ab, the first term of the multiplicand, by abc, and work to the right.

EXERCISE 21.

Multiply:

1. $5a+3b$ by $2a^2$.
2. $ab-bc$ by $5a^3bc$.
3. $ab-ac-bc$ by abc.
4. $6a^5b-7a^2b^2c$ by a^2b^2c.
5. $x-y-z$ by $-3x^5y^7z^9$.
6. x^2+2y^2-z by $-3x^2$.
7. $a^2+b^2-c^2$ by a^5bc^2.
8. $5a^2-3b^2+2c^2$ by $4ab^3c^2$
9. $abc-3a^3bc^2$ by $-2ab^2c$.
10. $xyz^2+x^2y^3z$ by $-x^2yz$.
11. $3x-2y-4$ by $5x^2$.
12. $3x^2-4y^2+5z^2$ by $2x^2y$
13. $a^3x-5a^2x^2+ax^3+2x^4$ by ax^2y.
14. $-9a^5+3a^3b^2-4a^2b^3-b^5$ by $-3ab^4$.
15. $3x^3-2x^2y-7xy^2+y^3$ by $-5x^2y$.
16. $-4xy^2+5x^2y+8x^3$ by $-3x^2y$.
17. $-3+2ab+a^2b^2$ by $-a^4$.
18. $-z-2xz^2+5x^2yz^2$ by $-3x^3yz$.

Multiplication of Polynomials by Polynomials.

If we have $m + n + p$ to be multiplied by $a + b + c$, we may substitute M for the multiplicand $m + n + p$. Then

$$(a + b + c) M = aM + bM + cM. \qquad (\S\ 41)$$

If now we substitute for M its value $m + n + p$, we have

$$\begin{aligned} &aM + bM + cM \\ &\qquad = a(m + n + p) + b(m + n + p) + c(m + n + p) \\ &\qquad = am + an + ap + bm + bn + bp + cm + cn + cp. \end{aligned}$$

102. To Find the Product of Two Polynomials, therefore,

Multiply every term of the multiplicand by each term of the multiplier, and add the partial products.

In multiplying polynomials, it is a convenient arrangement to write the multiplier under the multiplicand, and place like terms of the partial products in columns.

1. Multiply $5a - 6b$ by $3a - 4b$.

$$\begin{array}{rrr} 5a & -\ 6b & \\ 3a & -\ 4b & \\ \hline 15a^2 & -\ 18ab & \\ & -\ 20ab & +\ 24b^2 \\ \hline 15a^2 & -\ 38ab & +\ 24b^2 \end{array}$$

We multiply $5a$, the first term of the multiplicand, by $3a$, the first term of the multiplier, and obtain $15a^2$; then we multiply $-6b$, the second term of the multiplicand, by $3a$, the first term of the multiplier, and obtain $-18ab$. The first line of partial products is $15a^2 - 18ab$. In multiplying by $-4b$, we obtain for a second line of partial products $-20ab + 24b^2$, and this is put one place to the right, so that the like terms $-18ab$ and $-20ab$ may stand in the same column. We then add the coefficients of the like terms, and obtain the complete product in its simplest form.

2. Multiply $4x + 3 + 5x^2 - 6x^3$ by $4 - 6x^2 - 5x$.

Arrange both multiplicand and multiplier according to the ascending powers of x.

$$\begin{array}{rrrrrr}
3 & +\;4x & +\;5x^2 & -\;6x^3 & & \\
4 & -\;5x & -\;6x^2 & & & \\
\hline
12 & +16x & +20x^2 & -24x^3 & & \\
 & -15x & -20x^2 & -25x^3 & +30x^4 & \\
 & & -18x^2 & -24x^3 & -30x^4 & +36x^5 \\
\hline
12 & +\;x & -18x^2 & -73x^3 & & +36x^5
\end{array}$$

3. Multiply $1 + 2x + x^4 - 3x^2$ by $x^3 - 2 - 2x$.

Arrange according to the descending powers of x.

$$\begin{array}{rrrrrrr}
x^4 & -3x^2 & +2x & +1 & & & \\
x^3 & -2x & -2 & & & & \\
\hline
x^7 & -3x^5 & +2x^4 & +\;x^3 & & & \\
 & -2x^5 & & +6x^3 & -4x^2 & -2x & \\
 & & -2x^4 & & +6x^2 & -4x & -2 \\
\hline
x^7 & -5x^5 & & +7x^3 & +2x^2 & -6x & -2
\end{array}$$

4. Multiply $a^2 + b^2 + c^2 - ab - bc - ac$ by $a + b + c$.

Arrange according to the descending powers of a.

$$\begin{array}{rrrrrrrrrr}
a^2 & -\;ab & -\;ac & +\;b^2 & -\;bc & +\;c^2 & & & & \\
a & +\;b & +\;c & & & & & & & \\
\hline
a^3 & -a^2b & -a^2c & +ab^2 & -\;abc & +ac^2 & & & & \\
 & +a^2b & & -ab^2 & -\;abc & & +b^3 & -b^2c & +bc^2 & \\
 & & +a^2c & & -\;abc & -ac^2 & & +b^2c & -bc^2 & +c^3 \\
\hline
a^3 & & & & -3abc & & +b^3 & & & +c^3
\end{array}$$

Note. The student should observe that, with a view to bringing like terms of the partial products in columns, the terms of the multiplicand and multiplier are arranged in the *same order*.

Exercise 22.

Multiply:

1. $x + 10$ by $x + 6$.
2. $x - 2$ by $x - 3$.
3. $x - 3$ by $x + 5$.
4. $x + 3$ by $x - 3$.
5. $x - 11$ by $x - 1$.
6. $-x + 2$ by $-x - 3$.
7. $-x - 2$ by $x - 2$.
8. $-x + 4$ by $x - 4$.
9. $-x + 7$ by $x + 7$.
10. $x - 7$ by $x + 7$.
11. $x - 3$ by $2x + 3$.
12. $2x - 3$ by $x + 3$.
13. $x - 7$ by $2x - 1$.
14. $m - n$ by $2m + 1$.
15. $m - a$ by $m + a$.
16. $3x + 7$ by $2x - 3$.
17. $5x - 2y$ by $5x + 2y$.
18. $3x - 4y$ by $2x + 3y$.
19. $x^2 + y^2$ by $x^3 - y^3$.
20. $2x^2 + 3y^2$ by $x^2 + y^2$.
21. $x + y + z$ by $x - y + z$.
22. $x + 2y - z$ by $x - y + 2z$
23. $x^2 - xy + y^2$ by $x^2 + xy + y^2$.
24. $m^2 - mn + n^2$ by $m + n$.
25. $m^2 + mn + n^2$ by $m - n$.
26. $a^2 - 3ab + b^2$ by $a^2 - 3ab - b^2$.
27. $a^2 - 7a + 2$ by $a^2 - 2a + 3$.
28. $2x^2 - 3xy + 4y^2$ by $3x^2 + 4xy - 5y^2$.
29. $x^2 + xy + y^2$ by $x^2 - xz - z^2$.
30. $x^2 + y^2 + z^2 - xy - xz - yz$ by $x + y + z$.
31. $4a^2 - 10ab + 25a^2b^2$ by $5b + 2a$.
32. $x^2 + 4y$ by $y^2 + 4x$.
33. $x^2 + 2xy + 8$ by $y^2 + 2xy - 8$.
34. $a^2 + b^2 + 1 - ab - a - b$ by $a + 1 + b$.
35. $3x^2 - 2y^2 + 5z^2$ by $8x^2 + 2y^2 - 3z^2$.
36. $a^2 + b^2 + c^2 - ab - ac - bc$ by $a + b + c$
37. $x^2 - xy + y^2 + x + y + 1$ by $x + y - 1$.

38. $5x + 4x^2 + x^3 - 24$ by $x^2 + 11 - 4x$.
39. $x^3 + 11x - 4x^2 - 24$ by $x^2 + 5 + 4x$.
40. $x^4 + x^2 - 4x - 11 + 2x^3$ by $x^2 - 2x + 3$.
41. $-5x^4 - x^2 - x + x^5 + 13x^3$ by $x^2 - 2 - 2x$.
42. $3x + x^3 - 2x^2 - 4$ by $2x + 4x^3 + 3x^2 + 1$.
43. $5a^4 + 2a^2b^2 + ab^3 - 3a^3b$ by $5a^3b - 2ab^3 + 3a^2b^2 + b^4$.
44. $4a^7y - 32ay^4 - 8a^5y^2 + 16a^3y^3$ by $a^6y^2 + 4a^2y^4 + 4a^4y^3$.
45. $6a^5b + 3a^2b^4 - 2ab^5 + b^6$ by $4a^4 - 2ab^3 - 3b^4$.
46. $x^2 + y^2 + 2xy - 2x - 2y - 1$ by $x + y - 1$.
47. $a^m + 2a^{m-1} - 3a^{m-2} - 1$ by $a + 1$.
48. $a^n - 4a^{n-1} + 5a^{n-2} + a^{n-3}$ by $a - 1$.
49. $a^{4n+1} - 4a^{3n} + 2a^{2n-1} - a^{n-2}$ by $2a^3 - a^2 + a$.
50. $x^n - y^{n+1}$ by $x^n + y^{n+1}$.
51. $x^{2n} + 2x^ny^n + y^{2n}$ by $x^{2n} - 2x^ny^n + y^{2n}$.
52. $a^n - a^{n+1} - a^{n+2} + 1$ by $a^n + 1$.
53. $a^{3n} - a^{2n} + a^n - 1$ by $a^n - 1$.
54. $a^n - a^{n-1} + a^{n-2} + a^{n-3}$ by $a^2 - 1$.

EXERCISE 23. — REVIEW.

Simplify:

1. $(x^2 - x - 19)(x^2 + 2x - 3)$.
2. $(1 + 2x + x^2)(1 - x^3 + 2x^2 - 3x)$.
3. $(2x^2 + 2 + 3x)(2x - 3x^2 + 2x^3)$.
4. $(3x^2 + 5 - 4x)(8 + 6x^2 - 7x)$.
5. $(x^3 + x - y) - (x^2 - y^2 + xy)(x - 1)$.
6. $(3 + 7x^2 - 5x)(8x^2 - 6x - 10x^3 + 4)$.
7. $(b^3 + 6ab^2 - 4a^2b)(2a^2b - ab^2 - 8a^3)$.
8. $(x^2 + ax - b) - (x^2 + 5x - 4)(x - 3)$.
9. $(x^3 - mx^2 + nx + r)(x^2 + cx + d)$.
10. $\{x^2 - (a + b)x + ab\}(x - c)$.

11. $(x^3 + x^2y + xy^2 + y^3)(x - y)$.

12. $(4x^2 + 9y^2 - 6xy)(4x^2 + 9y^2 + 6xy)$

13. $(x^4 - 3x^2 + 5)(x^2 + 4)(x - 2)$.

14. $(x^4 - x^2y^2 + y^4)(x^4 + x^2y^2 + y^4)$.

15. $(2x^5 - 3x^3 + 4x^2 - 5)(x^2 - 8)(x - 3)$.

16. $(a^{2m} - a^m y^m + y^{2m})(a^m + y^m)(a + y)$.

17. $(a^{3x} - a^{2x} + a^x - 1)(a^x + 1)(a^x - 1)$.

18. $(a^2 + b^2 + c^2 + 2ab - ac - bc)(a + b + c)$.

19. $(a - 2b)(b - 2a) - (a - 3b)(4b - a) + 2ab$.

20. $(a + b + c)(a + b - c) - (2ab - c^2)$.

21. $(m + n)m - [(m - n)^2 - n(n - m)]$.

22. $[ac - (a - b)(b + c)] - b[b - (a - c)]$.

23. $(x - 1)(x - 2) - 3x(x + 3) + 2[(x + 2)(x + 1) - 3]$.

24. $4(a - 3b)(a + 3b) - 2(a - 6b)^2 - 2(a^2 + 6b^2)$.

25. $(x + y + z)^2 - x(y + z - x) - y(x + z - y) - z(x + y - z)$.

26. $5\{(a - b)x - cy\} - 2\{a(x - y) - bx\} - (3ax - 5cy)$.

27. $(a - b)x - (b - c)a - \{(b - x)(b - c) - (b - c)(b + c)\}$.

28. $(a + b)(b + c) - (c + d)(a + d) - (a + c)(b - d)$.

29. $a^2(b - c) - b^2(a - c) + c^2(a - b) - (a - b)(a - c)(b - c)$.

30. $(2a - b)^2 + 2b(a + b) - 3a^2 - (a - b)^2 + (a + b)(a - b)$

If $a = 0$, $b = 1$, $c = -1$, find the value of :

31. $(b - c)^2 + (c - a)^2 + (a - b)^2$.

32. $(b + c)^3 - (c - a)^2 - (a - b)$.

33. $(a - b)(a - c) + c(3a - b - c) + 2ac - (a - c) + 2b$.

34. $(b - c)^3 + (a - b)^3 + (c - a)^3 - 3(b - c)(a - b)(c - a)$.

35. If $a = 12$, and $b = 5$, find the values of

$$\sqrt{a^2} + \sqrt{b^2}; \text{ and } \sqrt{a^2 + b^2}.$$

Division of Compound Expressions.

Division of a Polynomial by a Monomial.

By the *distributive law*, (§ 43),

1. $$\frac{9a^4b^2x - 12a^3bx^2 - 3a^2x}{3a^2x} = \frac{9a^4b^2x}{3a^2x} - \frac{12a^3bx^2}{3a^2x} - \frac{3a^2x}{3a^2x}$$
$$= 3a^2b^2 - 4abx - 1.$$

2. $$\frac{6x^{4n+1} - 4x^{3n}}{2x^{2n-1}} = \frac{6x^{4n+1}}{2x^{2n-1}} - \frac{4x^{3n}}{2x^{2n-1}} = 3x^{2n+2} - 2x^{n+1}.$$

NOTE. Here we have $4n+1-(2n-1)=4n+1-2n+1=2n+2$, and $3n-(2n-1)=3n-2n+1=n+1$, as indices of x in the first and last terms of the quotient respectively.

103. To Divide a Polynomial by a Monomial, therefore,

Divide each term of the dividend by the divisor, and connect the partial quotients with their proper signs.

EXERCISE 24.

Divide:

1. $x^2 + 2xy$ by x.
2. $a^2 - 2ab$ by a.
3. $4x^6 - 8x^3$ by $2x^3$.
4. $-6x^3 - 2x$ by $-2x$.
5. $-8a^5 - 16a^{10}$ by $-8a^3$.
6. $27a^3 - 36a^2$ by $9a^2$.
7. $-30a^7 + 20a^9$ by $-10a^6$.
8. $-12x^6y^4 - 4x^2y^2$ by $-4x^2y^2$.
9. $-3x^7z^7 - 6x^5z^5$ by $-3x^3z^3$.
10. $3a^3b^3c^7 - 9a^6b^7c^5$ by $3a^3b^3c^5$.
11. $x^2 - xy - xz$ by $-x$.
12. $3a^3 - 6a^2b - 9ab^2$ by $-3a$.
13. $x^5y^2 - x^3y^3 - x^2y^2$ by x^2y^2.
14. $a^3b^2c - a^2b^3c - a^2bc^3$ by abc.
15. $8a^3 - 4a^2b - 6ab^2$ by $-2a$.
16. $5m^3n - 10m^2n^2 - 15mn^3$ by $5mn$.

Division of One Polynomial by Another.

$$\begin{array}{ll}
\text{If the divisor (one factor)} = & a + b + c, \\
\text{and the quotient (other factor)} = & \underline{n + p + q,} \\
\text{then the dividend (product)} = & \left\{\begin{array}{l} + an + bn + cn \\ + ap + bp + cp \\ + aq + bq + cq. \end{array}\right.
\end{array}$$

The first term of the dividend is an; that is, the product of a, the first term of the divisor, by n, the first term of the quotient. The first term n of the quotient is therefore found by dividing an, the first term of the dividend, by a, the first term of the divisor.

If the partial product formed by multiplying the entire divisor by n is subtracted from the dividend, the first term of the remainder ap is the product of a, the first term of the divisor, by p, the second term of the quotient; that is, the second term of the quotient is obtained by dividing the first term of the remainder by the first term of the divisor. In like manner, the third term of the quotient is obtained by dividing the first term of the new remainder by the first term of the divisor; and so on. Therefore,

104. We have the following rule for division:

Arrange both the dividend and divisor in ascending or descending powers of some common letter.

Divide the first term of the dividend by the first term of the divisor.

Write the result as the first term of the quotient.

Multiply all the terms of the divisor by the first term of the quotient.

Subtract the product from the dividend.

If there is a remainder, consider it as a new dividend and proceed as before.

105. It is of fundamental importance to arrange the dividend and divisor *in the same order* with respect to a common letter, and *to keep this order throughout the operation.*

The beginner should study carefully the processes in the following examples:

1. Divide $x^2 + 18x + 77$ by $x + 7$.

$$\begin{array}{r|l} x^2 + 18x + 77 & x + 7 \\ \cline{2-2} x^2 + 7x \phantom{{}+77} & x + 11 \\ \cline{1-1} 11x + 77 & \\ 11x + 77 & \\ \cline{1-1} \end{array}$$

Note. The student will notice that by this process we have in effect separated the dividend into two parts, $x^2 + 7x$ and $11x + 77$, and divided each part by $x + 7$, and that the complete quotient is the sum of the partial quotients x and 11. Thus,

$$x^2 + 18x + 77 = x^2 + 7x + 11x + 77 = (x^2 + 7x) + (11x + 77);$$
$$\therefore \frac{x^2 + 18x + 77}{x + 7} = \frac{x^2 + 7x}{x + 7} + \frac{11x + 77}{x + 7} = x + 11.$$

2. Divide $a^2 - 2ab + b^2$ by $a - b$.

$$\begin{array}{r|l} a^2 - 2ab + b^2 & a - b \\ \cline{2-2} a^2 - ab \phantom{{}+b^2} & a - b \\ \cline{1-1} -ab + b^2 & \\ -ab + b^2 & \\ \cline{1-1} \end{array}$$

3. Divide $4a^4x^2 - 4a^2x^4 + x^6 - a^6$ by $x^2 - a^2$.

Arrange according to descending powers of x.

$$\begin{array}{r|l} x^6 - 4a^2x^4 + 4a^4x^2 - a^6 & x^2 - a^2 \\ \cline{2-2} x^6 - a^2x^4 \phantom{{}+4a^4x^2 - a^6} & x^4 - 3a^2x^2 + a^4 \\ \cline{1-1} -3a^2x^4 + 4a^4x^2 - a^6 & \\ -3a^2x^4 + 3a^4x^2 \phantom{{}-a^6} & \\ \cline{1-1} a^4x^2 - a^6 & \\ a^4x^2 - a^6 & \\ \cline{1-1} \end{array}$$

4. Divide $22a^2b^2 + 15b^4 + 3a^4 - 10a^3b - 22ab^3$ by $a^2 + 3b^2 - 2ab$.

Arrange according to descending powers of a.

$$\begin{array}{l|l}
3a^4 - 10a^3b + 22a^2b^2 - 22ab^3 + 15b^4 & a^2 - 2ab + 3b^2 \\ \cline{2-2}
3a^4 - 6a^3b + 9a^2b^2 & 3a^2 - 4ab + 5b^2 \\ \cline{1-1}
\quad -4a^3b + 13a^2b^2 - 22ab^3 + 15b^4 & \\
\quad -4a^3b + 8a^2b^2 - 12ab^3 & \\ \cline{1-1}
\qquad\qquad 5a^2b^2 - 10ab^3 + 15b^4 & \\
\qquad\qquad 5a^2b^2 - 10ab^3 + 15b^4 & \\ \cline{1-1}
\end{array}$$

5. Divide $5x^3 - x + 1 - 3x^4$ by $1 + 3x^2 - 2x$.

Arrange according to ascending powers of x.

$$\begin{array}{l|l}
1 - x + 5x^3 - 3x^4 & 1 - 2x + 3x^2 \\ \cline{2-2}
1 - 2x + 3x^2 & 1 + x - x^2 \\ \cline{1-1}
\quad x - 3x^2 + 5x^3 - 3x^4 & \\
\quad x - 2x^2 + 3x^3 & \\ \cline{1-1}
\qquad - x^2 + 2x^3 - 3x^4 & \\
\qquad - x^2 + 2x^3 - 3x^4 & \\ \cline{1-1}
\end{array}$$

6. Divide $x^3 + y^3 + z^3 - 3xyz$ by $x + y + z$.

Arrange according to descending powers of x.

$$\begin{array}{l|l}
x^3 - 3xyz + y^3 + z^3 & x + y + z \\ \cline{2-2}
x^3 + x^2y + x^2z & x^2 - xy - xz + y^2 - yz + z^2 \\ \cline{1-1}
\quad - x^2y - x^2z - 3xyz + y^3 + z^3 & \\
\quad - x^2y - xy^2 - xyz & \\ \cline{1-1}
\qquad - x^2z + xy^2 - 2xyz + y^3 + z^3 & \\
\qquad - x^2z - xyz - xz^2 & \\ \cline{1-1}
\qquad\qquad xy^2 - xyz + xz^2 + y^3 + z^3 & \\
\qquad\qquad xy^2 + y^3 + y^2z & \\ \cline{1-1}
\qquad\qquad\quad - xyz + xz^2 - y^2z + z^3 & \\
\qquad\qquad\quad - xyz - y^2z - yz^2 & \\ \cline{1-1}
\qquad\qquad\qquad\quad xz^2 + yz^2 + z^3 & \\
\qquad\qquad\qquad\quad xz^2 + yz^2 + z^3 & \\ \cline{1-1}
\end{array}$$

7. Divide $4\,a^{x+1} - 30\,a^{x} + 19\,a^{x-1} + 5\,a^{x-2} + 9\,a^{x-4}$ by $a^{x-3} - 7\,a^{x-4} + 2\,a^{x-5} - 3\,a^{x-6}$.

$$\begin{array}{l|l}
4a^{x+1}-30a^{x}+19a^{x-1}+\ 5a^{x-2}+9a^{x-4} & a^{x-3}-7a^{x-4}+2a^{x-5}-3a^{x-6} \\ \cline{2-2}
\underline{4a^{x+1}-28a^{x}+\ 8a^{x-1}-12a^{x-2}\phantom{+9a^{x-4}}} & 4a^{4}-2a^{3}-3a^{2} \\
\qquad -\ 2a^{x}+11a^{x-1}+17a^{x-2}+9a^{x-4} & \\
\qquad \underline{-\ 2a^{x}+14a^{x-1}-\ 4a^{x-2}+6a^{x-3}} & \\
\qquad\qquad -\ 3a^{x-1}+21a^{x-2}-6a^{x-3}+9a^{x-4} & \\
\qquad\qquad \underline{-\ 3a^{x-1}+21a^{x-2}-6a^{x-3}+9a^{x-4}} &
\end{array}$$

Note. We find the index of a in the first term of the quotient by subtracting the index of a in the first term of the divisor from the index of a in the first term of the dividend. Now, $(x + 1) - (x - 3) = x + 1 - x + 3 = 4$. Hence, 4 is the index of a in the first term of the quotient. In the same way the other indices are found.

Exercise 25.

Divide:

1. $a^2 + 7\,a + 12$ by $a + 4$.
2. $a^2 - 5\,a + 6$ by $a - 3$.
3. $x^2 + 2\,xy + y^2$ by $x + y$.
4. $x^2 - 2\,xy + y^2$ by $x - y$.
5. $x^2 - y^2$ by $x - y$.
6. $4\,x^2 + 12\,x + 9$ by $2\,x + 3$.
7. $6\,x^2 - 11\,x + 4$ by $3\,x - 4$.
8. $8\,x^2 - 10\,ax - 3\,a^2$ by $4\,x + a$.
9. $3\,a^2 - 4\,a - 4$ by $2 - a$.
10. $a^3 - 8\,a - 3$ by $3 - a$.
11. $a^4 + 11\,a^2 - 12\,a - 5\,a^3 + 6$ by $3 + a^2 - 3\,a$.
12. $y^4 - 9\,y^2 + y^3 - 16\,y - 4$ by $y^2 + 4 + 4\,y$.
13. $36 + m^4 - 13\,m^2$ by $6 + m^2 + 5\,m$.
14. $1 - s - 3\,s^2 - s^5$ by $1 + 2\,s + s^2$.
15. $b^6 - 2\,b^3 + 1$ by $b^2 - 2\,b + 1$.
16. $x^4 + 2\,x^2y^2 + 9\,y^4$ by $x^2 - 2\,xy + 3\,y^2$.
17. $a^5 + b^5$ by $a^4 - a^3b + a^2b^2 - ab^3 + b^4$.
18. $1 + 5\,x^3 - 6\,x^4$ by $1 - x + 3\,x^2$.

19. $8x^2y^2 + 9y^4 + 16x^4$ by $4x^2 + 3y^2 - 4xy$.
20. $x^3 + y^3 + z^3 + 3x^2y + 3xy^2$ by $x + y + z$.
21. $a^3 + b^3 + c^3 - 3abc$ by $a + b + c$.
22. $x^3 + 8y^3 + z^3 - 6xyz$ by $x^2 + 4y^2 + z^2 - xz - 2xy - 2yz$.
23. $2x^2 - 3y^2 + xy - xz - 4yz - z^2$ by $2x + 3y + z$.
24. $x^2 - y^2 - 2yz - z^2$ by $x + y + z$.
25. $x^4 + x^2y^2 + y^4$ by $x^2 + xy + y^2$.
26. $x^4 - 9x^2 + 12x - 4$ by $x^2 + 3x - 2$.
27. $y^5 - 2y^4 - 6y^3 + 4y^2 + 13y + 6$ by $y^3 + 3y^2 + 3y + 1$.
28. $y^4 - 5y^2z^2 + 4z^4$ by $y^2 - yz - 2z^2$.
29. $x^2 - 4y^2 - 9z^2 + 12yz$ by $x + 2y - 3z$.
30. $x^5 - 41x - 120$ by $x^2 + 4x + 5$.
31. $x^4 - 3 + 5x + x^3 - 4x^2$ by $3 - 2x - x^2$.
32. $6 - 2x^4 + 10x^3 - 11x^2 + x$ by $4x - 3 - 2x^2$.
33. $1 - 6x^5 + 5x^6$ by $1 - 2x + x^2$.
34. $x^4 + 81 + 9x^2$ by $3x - x^2 - 9$.
35. $x^3 - y^3$ by $x^2 + xy + y^2$.
36. $x^6 + y^6$ by $x^2 + y^2$.
37. $x^4 + a^2x^2 + a^4$ by $x^2 - ax + a^2$.
38. $a^2 - 2b^2 + ab - 3c^2 + 7bc + 2ac$ by $3c + a - b$.
39. $ab + 2a^2 - 3b^2 - 4bc - ac - c^2$ by $c + 2a + 3b$.
40. $15a^4 + 10a^3x + 4a^2x^2 + 6ax^3 - 3x^4$ by $3a^2 + 2ax - x^2$.
41. $a^3 - 8b^3 - 1 - 6ab$ by $a - 2b - 1$.
42. $x^{3n} - 3x^{2n}y^n + 3x^ny^{2n} - y^{3n}$ by $x^n - y^n$.
43. $a^{m+n}b^n - 4a^{m+n-1}b^{2n} - 27a^{m+n-2}b^{3n} + 42a^{m+n-3}b^{4n}$
    by $a^m + 3a^{m-1}b^n - 6a^{m-2}b^{2n}$.

106. Integral expressions may have *fractional coefficients*, since an algebraic expression is integral if it has no *letter* in the denominator. The processes with fractional coefficients are precisely the same as with integral coefficients, as will be seen by the following examples worked out:

1. Add $\frac{1}{2}a^2 - \frac{1}{3}ab + \frac{1}{4}b^2$ and $\frac{5}{6}a^2 + \frac{2}{3}ab - \frac{3}{4}b^2$.

$$\begin{array}{l}
\frac{1}{2}a^2 - \frac{1}{3}ab + \frac{1}{4}b^2 \\
\frac{5}{6}a^2 + \frac{2}{3}ab - \frac{3}{4}b^2 \\
\hline
\frac{4}{3}a^2 + \frac{1}{3}ab - \frac{1}{2}b^2
\end{array}$$

2. From $\frac{1}{2}a^2 - \frac{1}{3}ab + \frac{1}{4}b^2$ take $\frac{1}{3}a^2 - \frac{1}{2}ab + \frac{5}{6}b^2$.

$$\begin{array}{l}
\frac{1}{2}a^2 - \frac{1}{3}ab + \ \ \frac{1}{4}b^2 \\
\frac{1}{3}a^2 - \frac{1}{2}ab + \ \ \frac{5}{6}b^2 \\
\hline
\frac{1}{6}a^2 + \frac{1}{6}ab - \frac{7}{12}b^2
\end{array}$$

3. Multiply $\frac{1}{2}a^2 - \frac{1}{3}ab + \frac{1}{4}b^2$ by $\frac{1}{2}a - \frac{2}{3}b$.

$$\begin{array}{lllll}
\frac{1}{2}a^2 & - \frac{1}{3}ab & + \frac{1}{4}b^2 & & \\
\frac{1}{2}a & - \frac{2}{3}b & & & \\
\hline
\frac{1}{4}a^3 & - \frac{1}{6}a^2b & + \frac{1}{8}ab^2 & & \\
 & - \frac{1}{3}a^2b & + \frac{2}{9}ab^2 & - \frac{1}{6}b^3 & \\
\hline
\frac{1}{4}a^3 & - \frac{1}{2}a^2b & + \frac{25}{72}ab^2 & - \frac{1}{6}b^3 &
\end{array}$$

4. Divide $\frac{2}{3}b^3 + \frac{11}{8}bd^2 - \frac{443}{270}b^2d - \frac{5}{12}d^3$ by $\frac{3}{2}b - \frac{5}{3}d$.

$$\begin{array}{llll|l}
\frac{2}{3}b^3 & - \frac{443}{270}b^2d & + \frac{11}{8}bd^2 & - \frac{5}{12}d^3 & \frac{3}{2}b - \frac{5}{3}d \\
\cline{5-5}
\frac{2}{3}b^3 & - \ \ \frac{20}{27}b^2d & & & \frac{4}{9}b^2 - \frac{3}{5}bd + \frac{1}{4}d^2 \\
\cline{1-4}
 & - \ \ \frac{9}{10}b^2d & + \frac{11}{8}bd^2 & - \frac{5}{12}d^3 & \\
 & - \ \ \frac{9}{10}b^2d & + \quad bd^2 & & \\
\cline{2-4}
 & & \frac{3}{8}bd^2 & - \frac{5}{12}d^3 & \\
 & & \frac{3}{8}bd^2 & - \frac{5}{12}d^3 & \\
\cline{3-4}
\end{array}$$

EXERCISE 26.

1. Add $\frac{1}{2}a^2b + \frac{1}{3}b^2c^4 + \frac{2}{15}$ and $-\frac{3}{10}a^2b - \frac{1}{5}b^2c^4 - \frac{4}{9}$.
2. From $\frac{5}{2}x^2 + 3ax - \frac{7}{3}a^2$ take $2x^2 - \frac{3}{2}ax - \frac{1}{2}a^2$.
3. From $\frac{1}{2}y - \frac{5}{2}a - \frac{3}{4}x + \frac{1}{3}b$ take $\frac{1}{3}y + \frac{11}{4}a - \frac{2}{3}x$.
4. Multiply $\frac{1}{3}c^2 - \frac{1}{4}c - \frac{1}{2}$ by $\frac{1}{3}c^2 - \frac{1}{4}c + \frac{1}{2}$.
5. Multiply $\frac{1}{2}x - \frac{1}{3}x^2 + \frac{1}{4}x^3$ by $\frac{1}{2}x + \frac{1}{3}x^2 + \frac{1}{4}x^3$.
6. Multiply $0.5m^4 - 0.4m^3n + 1.2m^2n^2 + 0.8mn^3 - 1.4n^4$ by $0.4m^2 - 0.6mn - 0.8n^2$.
7. Divide $\frac{9}{16}a^4 - \frac{7}{8}a^3b + \frac{19}{36}a^2b^2 + \frac{1}{6}ab^3$ by $\frac{3}{2}a + \frac{1}{3}b$.
8. Divide $-\frac{1}{8}d^5 + d^2 - \frac{41}{24}d^3 + \frac{5}{6}d^4$ by $-\frac{3}{4}d^2 + 2d$.

EXERCISE 27. — REVIEW.

1. Find the value of $x^3 + y^3 + z^3 - 3xyz$, if $x = 1$, $y = 2$, and $z = -3$.
2. Find the value of $\sqrt{2bc} - a$, and of $\sqrt{2bc - a}$, if $b = 8$, $c = 9$, and $a = 23$.
3. Add $a^2b - ab^2 + b^3$ and $a^3 - \frac{1}{2}a^2b + ab^2 - \frac{3}{4}b^3$.
4. Multiply $a^{2m} - a^mb^m + b^{2m}$ by $a^m + b^m$.
5. Multiply $4a^{2m+4} + 6a^{m+3} + 9a^2$ by $2a^{m+4} - 3a^3$.
6. Divide $x^3 + 8y^3 - 125z^3 + 30xyz$ by $x + 2y - 5z$.
7. Simplify $(x - a)^2 - (x - b)^2 - (a - b)(a + b - 3x)$.
8. Find the coefficient of x in the expression
$$x + a - 2[2a - b(c - x)].$$
9. Multiply $4x^{m+2n-1} - 7x^{2m-3n+2} + 5x^{2n+3m-2}$ by $5x^{2-m-2n}$.
10. Divide $c^2d^{3-2x} - c^4d^{4-2x} - c^{4+m}d^{3-x}$ by $c^{3-m}d^{2-2x}$.
11. Divide $m^3y^n - m^{1+x}y^{1+n} + m^{5-x}y^{n-4}$ by $m^{5-x}y^{n-4}$.

12. Divide $a^{1+3y} - a^3 + a^{2+5y}$ by a^{2-x+2y}.

13. Divide $x - x^{m-5n+4} + x^{2m}$ by x^{2-m-2n}.

14. Divide $y^p - y^{3-4m} + y^{4p+1}$ by y^{2p-m+1}.

15. Divide $2\,x^{3n} - 6\,x^{2n}y^n + 6\,x^ny^{2n} - 2\,y^{3n}$ by $x^n - y^n$.

16. Divide $x^3 - 2\,ax^2 + a^2x - abx - b^2x + a^2b + ab^2$ by $x^2 - ax + bx - ab$.

17. Divide $x^{4m} + x^{2m} + 1$ by $x^{2m} - x^m + 1$.

18. Divide $3\,x^{m+7} - 4\,x^{m+6} - 12\,x^{m+5} - 9\,x^{m+4}$ by $x^{m+4} - 3\,x^{m+3}$.

19. Divide $6\,x^{4m+5} - 13\,x^{3m+5} + 13\,x^{2m+5} - 13\,x^{m+5} - 5\,x^5$ by $2\,x^{2m} - 3\,x^m - 1$.

20. Divide $12\,a^{5n-3} - a^{4n-2} - 20\,a^{3n-1} + 19\,a^{2n} - 10\,a^{n+1}$ by $4\,a^{2n} - 3\,a^{n+1} + 2\,a^2$.

21. Arrange according to descending powers of x the following expression, and enclose the coefficient of each power in a parenthesis with a minus sign before each parenthesis except the first:

$$x^3 - 2\,bx - a^2x^2 - ax - ax^2 - cx - a^2x^3 - bcx.$$

22. Divide $1.2\,a^4x - 5.494\,a^3x^2 + 4.8\,a^2x^3 + 0.9\,ax^4 - x^5$ by $0.6\,ax - 2\,x^2$.

23. Multiply $\frac{1}{2}\,a^2 - \frac{1}{5}\,ab + \frac{1}{3}\,b^2$ by $\frac{1}{2}\,a + \frac{1}{5}\,b$.

24. Multiply $\frac{2}{3}\,a^2 + ab + \frac{3}{2}\,b^2$ by $\frac{1}{2}\,a - \frac{1}{3}\,b$.

25. Divide $\frac{1}{4}\,a^3 + \frac{1}{72}\,ab^2 + \frac{1}{12}\,b^3$ by $\frac{1}{2}\,a + \frac{1}{3}\,b$.

26. Subtract $\frac{1}{4}\,x^2 + \frac{1}{5}\,xy + \frac{1}{6}\,y^2$ from $\frac{1}{3}\,x^2 - \frac{1}{2}\,xy + \frac{1}{8}\,y^2$.

27. Subtract $x^2 + \frac{1}{3}\,xy - \frac{1}{2}\,y^2$ from $2\,x^2 - \frac{1}{2}\,xy + y^2$.

28. If $a = 8$, $b = 6$, $c = -4$, find the value of

$$\sqrt{a^2 + 2\,bc} + \sqrt{b^2 + ac} + \sqrt{c^2 + ab}.$$

CHAPTER VI.

SPECIAL RULES.

Multiplication.

107. Square of the Sum of Two Numbers.

$$\begin{aligned}(a+b)^2 &= (a+b)(a+b)\\ &= a(a+b)+b(a+b)\\ &= a^2+ab+ab+b^2\\ &= a^2+2ab+b^2.\end{aligned}$$

Hence,

RULE 1. *The square of the sum of two numbers is the sum of their squares plus twice their product.*

108. Square of the Difference of Two Numbers.

$$\begin{aligned}(a-b)^2 &= (a-b)(a-b)\\ &= a(a-b)-b(a-b)\\ &= a^2-ab-ab+b^2\\ &= a^2-2ab+b^2.\end{aligned}$$

Hence,

RULE 2. *The square of the difference of two numbers is the sum of their squares minus twice their product.*

109. Product of the Sum and Difference of Two Numbers.

$$\begin{aligned}(a+b)(a-b) &= a(a-b)+b(a-b)\\ &= a^2-ab+ab-b^2\\ &= a^2-b^2.\end{aligned}$$

Hence,

RULE 3. *The product of the sum and difference of two numbers is the difference of their squares.*

110. The following rule for raising a monomial to any required power will be useful in solving examples in multiplication:

Raise the numerical coefficient to the required power, and multiply the exponent of each letter by the exponent of the required power.

Thus the square of $7a^2b^6$ is $49a^4b^{12}$.

EXERCISE 28.

Write the product of:

1. $(x+y)^2$.
2. $(x-a)^2$.
3. $(x+2b)^2$.
4. $(3x-2c)^2$.
5. $(4y-5)^2$.
6. $(3a^2+4z^2)^2$.
7. $(x+y)(x-y)$.
8. $(4z-3)(4z+3)$.
9. $(3a^2+4b^2)(3a^2-4b^2)$.
10. $(3a-c)(3a-c)$.
11. $(x+7b^2)(x+7b^2)$.
12. $(ax+2by)(ax-2by)$.

111. If we are required to multiply $a+b+c$ by $a+b-c$, we may abridge the ordinary process as follows:

$$(a+b+c)(a+b-c) = \{(a+b)+c\}\{(a+b)-c\}$$

By Rule 3, $\quad = (a+b)^2 - c^2$

By Rule 1, $\quad = a^2 + 2ab + b^2 - c^2$.

If we are required to multiply $a+b-c$ by $a-b+c$, we may put the expressions in the following forms, and perform the operation:

$$(a+b-c)(a-b+c) = \{a+(b-c)\}\{a-(b-c)\}$$

By Rule 3, $\quad = a^2 - (b-c)^2$

By Rule 2, $\quad = a^2 - (b^2 - 2bc + c^2)$

By § 40, $\quad = a^2 - b^2 + 2bc - c^2$.

EXERCISE 29.

Find the product of:

1. $x + y + z$ and $x - y - z$.
2. $x - y + z$ and $x - y - z$.
3. $ax + by + 1$ and $ax + by - 1$.
4. $1 + x - y$ and $1 - x + y$.
5. $a + 2b - 3c$ and $a - 2b + 3c$.
6. $a^2 - ab + b^2$ and $a^2 + ab + b^2$.
7. $m^2 + mn + n^2$ and $m^2 - mn + n^2$.
8. $2 + x + x^2$ and $2 - x - x^2$.
9. $a^2 + a + 1$ and $a^2 - a + 1$.
10. $3x + 2y - z$ and $3x - 2y + z$.
11. $1 + x + y$ and $1 + x - y$.
12. $a^2 - 2ax + 4x^2$ and $a^2 + 2ax + 4x^2$.
13. $x^2 - 2xy + y^2$ and $x^2 + 2xy + y^2$.
14. $x - y + 13z^4$ and $x - y - 13z^4$.
15. $x^2 - 5y^2 - 7z^3$ and $x^2 - 5y^2 + 7z^3$.

112. Square of any Polynomial. If we put x for a, and $y + z$ for b, in the identity

$$(a + b)^2 = a^2 + 2ab + b^2,$$

we have

$$\{x + (y + z)\}^2 = x^2 + 2x(y + z) + (y + z)^2,$$

or

$$\begin{aligned}(x + y + z)^2 &= x^2 + 2xy + 2xz + y^2 + 2yz + z^2 \\ &= x^2 + y^2 + z^2 + 2xy + 2xz + 2yz.\end{aligned}$$

The complete product consists of the sum of the squares of the terms of the given expression and twice the product of each term into all the terms that follow it.

Again, if we put $a - b$ for a, and $c - d$ for b, in the same identity, we have

$$\begin{aligned}
&\{(a - b) + (c - d)\}^2 \\
&\quad = (a - b)^2 + 2(a - b)(c - d) + (c - d)^2 \\
&\quad = (a^2 - 2ab + b^2) + 2a(c - d) - 2b(c - d) + (c^2 - 2cd + d^2) \\
&\quad = a^2 - 2ab + b^2 + 2ac - 2ad - 2bc + 2bd + c^2 - 2cd + d^2 \\
&\quad = a^2 + b^2 + c^2 + d^2 - 2ab + 2ac - 2ad - 2bc + 2bd - 2cd.
\end{aligned}$$

Here the same law holds as before, the sign of each double product being + or −, according as the factors composing it have *like* or *unlike* signs. The same is true for any polynomial. Hence we have the following rule:

Rule 4. *The square of a polynomial is the sum of the squares of the several terms and twice the product obtained by multiplying each term into all the terms that follow it.*

Exercise 30.

Write the square of:

1. $2x - 3y$.
2. $a + b + c$.
3. $x + y - z$.
4. $x - y + z$.
5. $x + y + 5$.
6. $x + 2y + 3$.
7. $a - b + c$.
8. $3x - 2y + 4$.
9. $2x - 3y + 4z$.
10. $x^2 + y^2 + z^2$.
11. $2x - y - z$.
12. $a - 2b - 3c$.
13. $3a - b + 2c$.
14. $x + 2y - 3z$.
15. $x^2 - y^2 + z^2$
16. $x - 2y - 3z$.
17. $2z - y + x$.
18. $x + y + z + 1$.
19. $x - y + z - 1$.
20. $4x + y + z - 2$.
21. $2x - y - z - 3$.
22. $x - 2y - 3z + 4$.

113. Product of Two Binomials. The product of two binomials which have the form $x + a$, $x + b$, should be carefully noticed and remembered.

1. $(x+5)(x+3) = x(x+3) + 5(x+3)$
$= x^2 + 3x + 5x + 15$
$= x^2 + 8x + 15.$

2. $(x-5)(x-3) = x(x-3) - 5(x-3)$
$= x^2 - 3x - 5x + 15$
$= x^2 - 8x + 15.$

3. $(x+5)(x-3) = x(x-3) + 5(x-3)$
$= x^2 - 3x + 5x - 15$
$= x^2 + 2x - 15.$

4. $(x-5)(x+3) = x(x+3) - 5(x+3)$
$= x^2 + 3x - 5x - 15$
$= x^2 - 2x - 15.$

Each of these results has three terms.

The first term of each result is the product of the first terms of the binomials.

The last term of each result is the product of the second terms of the binomials.

The middle term of each result has for the coefficient of x the *algebraic sum* of the second terms of the binomials.

The intermediate step given above may be omitted, and the products written at once by *inspection*. Thus,

1. Multiply $x + 8$ by $x + 7$.
$8 + 7 = 15;\ 8 \times 7 = 56.$
$\therefore (x+8)(x+7) = x^2 + 15x + 56.$

2. Multiply $x - 8$ by $x - 7$.

$$(-8) + (-7) = -15; \quad (-8)(-7) = +56.$$
$$\therefore (x-8)(x-7) = x^2 - 15x + 56.$$

3. Multiply $x - 7y$ by $x + 6y$.

$$-7y + 6y = -y; \quad (-7y)(6y) = -42y^2.$$
$$\therefore (x-7y)(x+6y) = x^2 - xy - 42y^2.$$

4. Multiply $x^2 + 6(a+b)$ by $x^2 - 5(a+b)$.

$$-5(a+b) + 6(a+b) = (a+b);$$
$$-5(a+b) \times 6(a+b) = -30(a+b)^2.$$
$$\therefore \{x^2+6(a+b)\}\{x^2-5(a+b)\} = x^4+(a+b)x^2-30(a+b)^2.$$

EXERCISE 31.

Find by inspection the product of:

1. $(x+8)(x+3)$.
2. $(x+8)(x-3)$.
3. $(x-7)(x+10)$.
4. $(x-9)(x-5)$.
5. $(x-10)(x+9)$.
6. $(a-10)(a-5)$.
7. $(x-3a)(x+2a)$.
8. $(a+2b)(a-4b)$.
9. $(a-12)(a-3)$.
10. $(a+2b)(a+4b)$.
11. $(a-3b)(a+7b)$.
12. $(a+2b)(a-9b)$.
13. $(x-3a)(x-4a)$.
14. $(x+4z)(x-2z)$.
15. $(x+6y)(x-5y)$.
16. $(x^2-9)(x^2+8)$.
17. $(x^2+2y^2)(x^2-3y^2)$.
18. $(x^2+8y^2)(x^2-4y^2)$.
19. $(ab-8)(ab+5)$.
20. $(ab-7xy)(ab+3xy)$.
21. $(x-3y)(x-3y)$.
22. $(x+6)(x+6)$.
23. $(a-3b)(a-3b)$.
24. $(x-c)(x-d)$.
25. $(x+a)(x-b)$.
26. $(x-a)(x+b)$.
27. $\{(a+b)+2\}\{(a+b)-4\}$.
28. $\{(x+y)-2\}\{(x+y)+4\}$.

114. In like manner the product of *any* two binomials may be written.

1. Multiply $2a - b$ by $3a + 4b$.

$$\begin{aligned}(2a - b)(3a + 4b) &= 6a^2 + 8ab - 3ab - 4b^2\\ &= 6a^2 + 5ab - 4b^2.\end{aligned}$$

2. Multiply $2x + 3y$ by $3x - 2y$.

The middle term is

$$2x \times (-2y) + 3y \times 3x = 5xy.$$
$$\therefore (2x + 3y)(3x - 2y) = 6x^2 + 5xy - 6y^2.$$

Exercise 32.

Find the product of:

1. $3x - y$ and $2x + y$.
2. $4x - 3y$ and $3x - 2y$.
3. $5x - 4y$ and $3x - 4y$.
4. $x - 7y$ and $2x - 5y$.
5. $11x - 2y$ and $7x + y$.
6. $10x - 3y$ and $10x - 7y$.
7. $3a^2 - 2b^2$ and $2a^2 + 3b^2$.
8. $a^2 + b^2$ and $a - b$.
9. $3a^2 - 2b^2$ and $2a + 3b$.
10. $a^2 - b^2$ and $a + b$.
11. $4x + 1$ and $3x - 2$.
12. $3x - 5$ and $x + 1$.
13. $3a^2 + x^2$ and $4a^2 - x^2$.
14. $2x + y$ and $x + 2y$.
15. $3b + x$ and $2b - 3x$.
16. $2a + 5b$ and $4a - 3b$.
17. $4x + 3y$ and $2x - 7y$.
18. $2y + 3z$ and $3y - z$.
19. $2x + 7y$ and $3x - y$.
20. $3a - 2c$ and $2a - 5c$.

Division.

115. The following rule for finding any required root of a monomial will be found useful in solving examples in division:

Find the required root of the numerical coefficient, and divide the exponent of each letter by the index of the required root.

Thus, the square root of $25x^2y^4$ is $5xy^2$.

116. Difference of Two Squares.

$$\frac{a^2-b^2}{a+b}=a-b;\quad \frac{a^2-b^2}{a-b}=a+b.\quad \text{Hence,}$$

RULE 1. *The difference of the squares of two numbers is divisible by the sum of the numbers, and the quotient is the difference of the numbers.*

The difference of the squares of two numbers is divisible by the difference of the numbers, and the quotient is the sum of the numbers.

EXERCISE 33.

Write by inspection the quotient of:

1. $\dfrac{a^2-4}{a-2}$.
2. $\dfrac{9-x^2}{3+x}$.
3. $\dfrac{16-a^2}{4+a}$.
4. $\dfrac{x^2-25}{x-5}$.
5. $\dfrac{36-x^2}{6+x}$.
6. $\dfrac{9a^2-b^2}{3a-b}$
7. $\dfrac{9a^4-25y^4}{3a^2+5y^2}$.
8. $\dfrac{4x^{16}-9y^6}{2x^8-3y^3}$.
9. $\dfrac{4x^{10}-a^8}{2x^5-a^4}$.
10. $\dfrac{a^2b^6c^8-x^{12}}{ab^3c^4+x^6}$.
11. $\dfrac{x^4a^8-b^{10}}{x^2a^4-b^5}$.
12. $\dfrac{a^2-(b+c)^2}{a-(b+c)}$.

117. Sum and Difference of Two Cubes. By performing the division, we find that

$$\frac{a^3+b^3}{a+b}=a^2-ab+b^2;\quad \frac{a^3-b^3}{a-b}=a^2+ab+b^2.$$ Hence,

RULE 2. *The* ***sum of the cubes*** *of two numbers is divisible by the sum of the numbers, and the quotient is the sum of the squares of the numbers* ***minus*** *their product.*

RULE 3. *The* ***difference of the cubes*** *of two numbers is divisible by the difference of the numbers, and the quotient is the sum of the squares of the numbers* ***plus*** *their product.*

EXERCISE 34.

Write by inspection the quotient of:

1. $\dfrac{1-8x^3}{1-2x}$.
2. $\dfrac{1+8x^3}{1+2x}$.
3. $\dfrac{27a^3-b^3}{3a-b}$.
4. $\dfrac{27a^3+b^3}{3a+b}$.
5. $\dfrac{64x^3+27y^3}{4x+3y}$.
6. $\dfrac{64x^3-27y^3}{4x-3y}$.
7. $\dfrac{1-27z^3}{1-3z}$.
8. $\dfrac{1+27z^3}{1+3z}$.
9. $\dfrac{a^3b^3-c^3}{ab-c}$.
10. $\dfrac{a^3b^3+c^3}{ab+c}$.
11. $\dfrac{64+y^3}{4+y}$.
12. $\dfrac{343-8a^3}{7-2a}$.
13. $\dfrac{8a^3+b^6}{2a+b^2}$.
14. $\dfrac{x^6+729y^3}{x^2+9y}$.
15. $\dfrac{a^6-27b^3}{a^2-3b}$.
16. $\dfrac{8x^3-64y^6}{2x-4y^2}$.

118. Sum and Difference of any Two Like Powers. By performing the division, we find that

$$\frac{a^4 - b^4}{a - b} = a^3 + a^2b + ab^2 + b^3;$$

$$\frac{a^4 - b^4}{a + b} = a^3 - a^2b + ab^2 - b^3;$$

$$\frac{a^5 - b^5}{a - b} = a^4 + a^3b + a^2b^2 + ab^3 + b^4;$$

$$\frac{a^5 + b^5}{a + b} = a^4 - a^3b + a^2b^2 - ab^3 + b^4.$$

We find by trial that

$a^2 + b^2$, $a^4 + b^4$, $a^6 + b^6$, and so on

are *not* divisible by $a + b$ or by $a - b$. Hence,

When n is a positive integer, it is proved in chap. vii,

1. *$a^n + b^n$ is divisible by $a + b$ if n is odd, and by neither $a + b$ nor $a - b$ if n is even.*
2. *$a^n - b^n$ is divisible by $a - b$ if n is odd, and by both $a + b$ and $a - b$ if n is even.*

NOTE. It is important to notice in the above examples that the terms of the quotient are all *positive* when the divisor is $a - b$, and *alternately positive and negative* when the divisor is $a + b$; also, that the quotient is homogeneous, the exponent of a decreasing and of b increasing by 1 for each successive term.

EXERCISE 35.

Find the quotient of:

1. $\dfrac{x^6 - y^6}{x - y}$.
2. $\dfrac{x^6 - y^6}{x + y}$.
3. $\dfrac{x^4 - 1}{x - 1}$.
4. $\dfrac{x^4 - 1}{x + 1}$.
5. $\dfrac{x^4 - 16}{x - 2}$.
6. $\dfrac{x^5 - 32}{x - 2}$.
7. $\dfrac{x^5 + 32}{x + 2}$.
8. $\dfrac{1 - m^4}{1 - m}$.
9. $\dfrac{1 + m^5}{1 + m}$.

CHAPTER VII.

FACTORS.

119. Rational Expressions. An expression is *rational* if none of its terms contain square or other roots.

120. Factors of Rational and Integral Expressions. By factors of *a given integral number* in Arithmetic we mean integral numbers that will exactly divide the given number.

Likewise, by factors of *a rational and integral expression* in Algebra we mean rational and integral expressions that will exactly divide the given expression.

121. Factors of Monomials. The factors of a monomial may be found by inspection. Thus, the factors of $14a^2b$ are 7, 2, a, a, and b.

122. Factors of Polynomials. The form of a polynomial that can be resolved into factors often suggests the process of finding the factors.

123. When the terms have a common monomial factor.

Resolve into factors $2x^2 + 6xy$.

Since 2 and x are factors of each term, we have

$$\frac{2x^2 + 6xy}{2x} = \frac{2x^2}{2x} + \frac{6xy}{2x} = x + 3y.$$

$$\therefore 2x^2 + 6xy = 2x(x + 3y).$$

Hence, the required factors are $2x$ and $x + 3y$.

EXERCISE 36.

Resolve into factors:

1. $3x^2 - 6x^3$.
2. $2a^2 - 4a$.
3. $5ab - 5a^2b^3$.
4. $3a^2b - 4ab^2$.
5. $8x^3y^2 + 4x^2y^3$.
6. $3a^3 - a^2 + a$.
7. $x^3 + x^2y - xy^2$.
8. $a^4 - a^3b + a^2b^2$.
9. $3x^4 - 9x^2 - 6x^3$.
10. $ab^2 - bc^2 + bx$.
11. $8a^2b - 6a^3 + 4ab$.
12. $4x^2y - 8xy^2 - 4xy$.
13. $8a^2x^2 - 4a^2b + 12a^2y^2$.
14. $8a^3b^2c^2 - 4a^2b^3c^3 + 2a^2b^2c^3$.
15. $15a^3x - 10a^3y + 5a^3z$.
16. $a^3cy^3 + 2a^2c^2y^2 - a^2cy^4$.
17. $3b^3c^3 + 2b^2c^2 - 6bc^3$.
18. $6a^2b - 3ab - 12ab^2$.
19. $5a^2c^3 + 3a^2c + 4a^2c^2$.
20. $6x^3y^2 + 3x^2y^2 - 15xy^3$.
21. $7a^2b^2c^2 - 14abc + 7ab^2c^3$.
22. $8x^2y^2 + 16xyz - 24x^2y^2z^2$.
23. $ab^2c^3 - 2a^2bc + 3a^3b^3c^2$.
24. $x^2y^2z^2 - x^3y^2z^3 + x^2y^3z$.

124. When the terms can be grouped so as to show a common compound factor.

1. Resolve into factors $ac + ad + bc + bd$.

$$\begin{aligned} ac + ad + bc + bd &= (ac + ad) + (bc + bd) && (1)\\ &= a(c + d) + b(c + d) && (2)\\ &= (a + b)(c + d). && (3) \end{aligned}$$

Since one factor is seen in (2) to be $c + d$, dividing by $c + d$ we obtain the other factor, $a + b$.

2. Resolve into factors $3x^2 + 6ax + bx + 2ab$.

$$\begin{aligned} 3x^2 + 6ax + bx + 2ab &= (3x^2 + 6ax) + (bx + 2ab)\\ &= 3x(x + 2a) + b(x + 2a)\\ &= (3x + b)(x + 2a). \end{aligned}$$

3. Find the factors of $ac + ad - bc - bd$.

$$\begin{aligned} ac + ad - bc - bd &= (ac + ad) - (bc + bd) \\ &= a(c + d) - b(c + d) \\ &= (a - b)(c + d). \end{aligned}$$

NOTE. Here the last two terms, $-bc - bd$, being put within a parenthesis preceded by the sign $-$, have their signs changed to $+$.

4. Resolve into factors $3x^3 - 5x^2 - 6x + 10$.

$$\begin{aligned} 3x^3 - 5x^2 - 6x + 10 &= (3x^3 - 5x^2) - (6x - 10) \\ &= x^2(3x - 5) - 2(3x - 5) \\ &= (x^2 - 2)(3x - 5). \end{aligned}$$

5. Resolve into factors $5x^3 - 15ax^2 - x + 3a$.

$$\begin{aligned} 5x^3 - 15ax^2 - x + 3a &= (5x^3 - 15ax^2) - (x - 3a) \\ &= 5x^2(x - 3a) - 1(x - 3a) \\ &= (5x^2 - 1)(x - 3a). \end{aligned}$$

6. Resolve into factors $6y - 27x^2y - 10x + 45x^3$.

$$\begin{aligned} 6y - 27x^2y - 10x + 45x^3 &= 6y - 10x - 27x^2y + 45x^3 \\ &= (6y - 10x) - (27x^2y - 45x^3) \\ &= 2(3y - 5x) - 9x^2(3y - 5x) \\ &= (2 - 9x^2)(3y - 5x). \end{aligned}$$

EXERCISE 37.

Resolve into factors:

1. $ax - bx + ay - by$.
2. $ax - bx - ay + by$.
3. $ax - cy - ay + cx$.
4. $x^2 + ax - bx - ab$.
5. $x^2 + xy - ax - ay$.
6. $x^2 - xy - 6x + 6y$.
7. $2ab - 3ac - 2by + 3cy$.
8. $2x^2 - 3xy + 4ax - 6ay$.
9. $ab - 3bc - 2ac + 6c^2$.
10. $x^3 + x - x^2z - z$.
11. $x^3 + 4x^2 + 3x + 12$.
12. $3ac - 3ax - c + x$.

13. $a^2b - abx - ac + cx$.	23. $a^2bx + b^2cx - a^2cy - bc^2y$.
14. $2x^3 - 3x^2 - 4x + 6$.	24. $3x^2 - 5y^2 - 6x^3 + 10xy^2$.
15. $ax^4 + bx^3 - ax - b$.	25. $8ax - 10bx - 12a + 15b$.
16. $ax^2 + a^2x + a + x$.	26. $6x^4 + 8x^3 - 9x^2 - 12x$.
17. $(x - y)^2 + 2y(x - y)$.	27. $3cx^4 - 2dx^3 - 9cx^2 + 6dx$.
18. $1 + 15x^4 - 5x - 3x^3$.	28. $(a + b)(c + d) - 3c(a + b)$.
19. $x^2 - x^3 + 1 - x$.	29. $b^3x + bc^2x - b^2cy - c^3y$.
20. $x^3 - 5x^2 + 2x - 10$.	30. $2ac - bc + 4a^2 - 2ab$.
21. $x^3 + 7x^2 + 3x + 21$.	31. $a^2n + ac^2 - amn - c^2m$.
22. $z^4 - z^3 - z + 1$.	32. $1 + c - c^2xy - c^3xy$.

125. When a trinomial is a perfect square. A trinomial is a perfect square if its first and last terms are perfect squares and positive, and its middle term is *twice the product of the square roots of the first and last terms.*

Thus, $16a^2 - 24ab + 9b^2$ is a perfect square.

The rule for extracting the square root of a perfect trinomial square is as follows:

Extract the square roots of the first and last terms, and connect these square roots by the sign of the middle term.

Thus, if we wish to find the square root of

$$16a^2 - 24ab + 9b^2,$$

we take the square roots of $16a^2$ and $9b^2$, which are $4a$ and $3b$, respectively, and connect these square roots by the sign of the middle term. The square root is therefore

$$4a - 3b.$$

In like manner, the square root of

$$16a^2 + 24ab + 9b^2 \text{ is}$$

$$4a + 3b.$$

127. If the squares are compound expressions, the same method may be employed.

1. Resolve into factors $(x + 3y)^2 - 16a^2$.

The square root of the first term is $x + 3y$.
The square root of the second term is $4a$.
The sum of these roots is $x + 3y + 4a$.
The difference of these roots is $x + 3y - 4a$.
Therefore, $(x + 3y)^2 - 16a^2 = (x + 3y + 4a)(x + 3y - 4a)$.

2. Resolve into factors $a^2 - (3b - 5c)^2$.

The square roots of the terms are a and $(3b - 5c)$.
The sum of these roots is $a + (3b - 5c)$, or $a + 3b - 5c$.
The difference of these roots is $a - (3b - 5c)$, or $a - 3b + 5c$.
Therefore, $a^2 - (3b - 5c)^2 = (a + 3b - 5c)(a - 3b + 5c)$.

If the factors have like terms, these terms should be collected so as to give the results in the simplest form.

3. Resolve into factors $(3a + 5b)^2 - (2a - 3b)^2$.

The square roots of the terms are $3a + 5b$ and $2a - 3b$.
The sum of these roots is $(3a + 5b) + (2a - 3b)$,
or $3a + 5b + 2a - 3b = 5a + 2b$.
The difference of these roots is $(3a + 5b) - (2a - 3b)$,
or $3a + 5b - 2a + 3b = a + 8b$.
Therefore, $(3a + 5b)^2 - (2a - 3b)^2 = (5a + 2b)(a + 8b)$.

Exercise 40.

Resolve into factors:

1. $(x + y)^2 - z^2$.
2. $(x - y)^2 - z^2$.
3. $(x - 2y)^2 - 4z^2$.
4. $(a + 3b)^2 - 16c^2$.
5. $(a - b)^2 - (c - d)^2$.
6. $(2a + b)^2 - 25c^2$.
7. $(x + 2y)^2 - (2x - y)^2$.
8. $(x + 3)^2 - (3x - 4)^2$.

9. $x^2 - (y - z)^2$.
10. $a^2 - (3b - 2c)^2$.
11. $b^2 - (2a + 3c)^2$.
12. $1 - (x + 5b)^2$.
13. $9a^2 - (x - 3c)^2$.
14. $16a^2 - (2y - 3z)^2$.
15. $(a + b - c)^2 - (a - b - c)^2$.
16. $(a - 3x)^2 - (3a - 2x)^2$.
17. $(2a - 1)^2 - (3a + 1)^2$.
18. $(x - 5)^2 - (x + y - 5)^2$.
19. $(2a + b - c)^2 - (a - 2b + c)^2$.
20. $(a + 2b - 3c)^2 - (a + 5c)^2$.

128. By properly grouping the terms, compound expressions may often be written as the difference of two squares, and the factors readily found.

1. Resolve into factors $a^2 - 2ab + b^2 - 9c^2$.

$$\begin{aligned} a^2 - 2ab + b^2 - 9c^2 &= (a^2 - 2ab + b^2) - 9c^2 \\ &= (a - b)^2 - 9c^2 \\ &= (a - b + 3c)(a - b - 3c). \end{aligned}$$

2. Resolve into factors $12ab + 9x^2 - 4a^2 - 9b^2$.

Here $12ab$ shows that it is the middle term of the expression which has in its first and last terms a^2 and b^2, and the minus sign before $4a^2$ and $9b^2$ shows that these terms must be put in a parenthesis with the minus sign before it, in order that they may be made positive.

Therefore, the arrangement will be

$$\begin{aligned} 9x^2 - (4a^2 - 12ab + 9b^2) &= 9x^2 - (2a - 3b)^2 \\ &= (3x + 2a - 3b)(3x - 2a + 3b). \end{aligned}$$

3. Resolve into factors $-a^2 + b^2 - c^2 + d^2 + 2ac + 2bd$.

Here $2bd$ must be grouped with b^2 and d^2, and $2ac$ with $-a^2$ and $-c^2$; and this last group put in a parenthesis preceded by $-$.

$$\begin{aligned} &-a^2 + b^2 - c^2 + d^2 + 2ac + 2bd \\ &= (b^2 + 2bd + d^2) - (a^2 - 2ac + c^2) \\ &= (b + d)^2 - (a - c)^2 \\ &= (b + d + a - c)(b + d - a + c). \end{aligned}$$

EXERCISE 41.

Resolve into factors:

1. $a^2 + 2ab + b^2 - 4c^2$.
2. $x^2 - 2xy + y^2 - 9a^2$.
3. $b^2 - x^2 + 4ax - 4a^2$.
4. $4a^2 + 4ab + b^2 - x^2$.
5. $a^2 - x^2 - y^2 - 2xy$.
6. $1 - a^2 - 2ab - b^2$.
7. $a^2 + b^2 + 2ab - 16a^2b^2$.
8. $4x^2 - 9a^2 + 6a - 1$.
9. $a^2 + b^2 - c^2 - d^2 - 2ab - 2cd$.
10. $x^2 + y^2 - 2xy - 2ab - a^2 - b^2$.
11. $9x^2 - 6x + 1 - a^2 - 4ab - 4b^2$.
12. $a^2 + 2ab - x^2 - 6xy - 9y^2 + b^2$.
13. $x^2 - 2x + 1 - b^2 + 2by - y^2$.
14. $9 - 6x + x^2 - a^2 - 8ab - 16b^2$.
15. $4 - 4x + x^2 - 4ab - b^2 - 4a^2$.
16. $a^4 - a^2 - 9 + b^4 + 6a - 2a^2b^2$.
17. $4a^2 + 9c^2 - 12ac + 12bd - 9b^2 - 4d^2$.
18. $4x^2 - 4y^2 - 4x + 1 - z^2 + 4yz$.
19. $4xy - x^2 + 1 - 4y^2$.
20. $c^2 - 2ac - 1 + 2b + a^2 - b^2$.
21. $4ac - 1 - 6x - 9x^2 + a^2 + 4c^2$.
22. $4 - 9x^2 - 4y^2 + 12xy$.
23. $4x^4 - 12x^2 - 9z^2 + 12yz - 4y^2 + 9$.
24. $a^4 - b^2 - 4x^4 - 6a^2c + 4bx^2 + 9c^2$.
25. $25a^2 - 1 - 10ab - 9a^2x^2 + b^2 + 6ax$.
26. $16x^4 + 30x^3 + 8c^2x^2 - 25x^6 - 9 + c^4$.
27. $4a^4 + 9b^6 - 12a^2b^3 - 81x^2y^4z^6$.

129. A trinomial in the form $a^4 + a^2b^2 + b^4$ can be written as the difference of two squares.

Since a trinomial is a perfect square when the middle term is *twice* the product of the square roots of the first and last terms, it is obvious that we must add a^2b^2 to the middle term of $a^4 + a^2b^2 + b^4$ to make it a perfect square. We must also subtract a^2b^2 to keep the value of the expression unchanged.

1. Resolve into factors $a^4 + a^2b^2 + b^4$.

$$\begin{aligned} a^4 + a^2b^2 + b^4 &= a^4 + 2\,a^2b^2 + b^4 - a^2b^2 \\ &= (a^2 + b^2)^2 - a^2b^2 \\ &= (a^2 + b^2 + ab)\,(a^2 + b^2 - ab) \\ &= (a^2 + ab + b^2)\,(a^2 - ab + b^2). \end{aligned}$$

2. Resolve into factors $4\,x^4 - 37\,x^2y^2 + 9\,y^4$.

Twice the product of the square roots of $4\,x^4$ and $9\,y^4$ is $12\,x^2y^2$. We may separate the term $-37\,x^2y^2$ into two terms, $-12\,x^2y^2$ and $-25\,x^2y^2$, and write the expression

$$\begin{aligned} &(4\,x^4 - 12\,x^2y^2 + 9\,y^4) - 25\,x^2y^2 \\ &\quad = (2\,x^2 - 3\,y^2)^2 - 25\,x^2y^2 \\ &\quad = (2\,x^2 - 3\,y^2 + 5\,xy)\,(2\,x^2 - 3\,y^2 - 5\,xy) \\ &\quad = (2\,x^2 + 5\,xy - 3\,y^2)\,(2\,x^2 - 5\,xy - 3\,y^2). \end{aligned}$$

Exercise 42.

Resolve into factors:

1. $x^4 + x^2y^2 + y^4$.
2. $x^4 + x^2 + 1$.
3. $9\,a^4 - 15\,a^2 + 1$.
4. $16\,a^4 - 17\,a^2 + 1$.
5. $4\,a^4 - 13\,a^2 + 1$.
6. $9\,a^4 + 26\,a^2b^2 + 25\,b^4$.
7. $4\,x^4 - 21\,x^2y^2 + 9\,y^4$.
8. $4\,a^4 - 29\,a^2c^2 + 25\,c^4$.
9. $4\,a^4 + 16\,a^2c^2 + 25\,c^4$.
10. $25\,x^4 + 31\,x^2y^2 + 16\,y^4$.

130. When a trinomial has the form $x^2 + ax + b$.

Where a is the *algebraic sum* of two numbers, and is either positive or negative; and b is the *product* of these two numbers, and is either positive or negative.

Since $(x + 5)(x + 3) = x^2 + 8x + 15$,
the factors of $x^2 + 8x + 15$ are $x + 5$ and $x + 3$.

Since $(x + 5)(x - 3) = x^2 + 2x - 15$,
the factors of $x^2 + 2x - 15$ are $x + 5$ and $x - 3$.

Hence, if a trinomial of the form $x^2 + ax + b$ is such an expression that it can be resolved into two binomial factors, the first term of each factor will be x; and the second terms of the factors will be two numbers *whose product is* b, the last term of the trinomial, and *whose algebraic sum is* a, the coefficient of x in the middle term of the trinomial.

1. Resolve into factors $x^2 + 11x + 30$.

We are required to find two numbers whose product is 30 and whose sum is 11.

Two numbers whose product is 30 are 1 and 30, 2 and 15, 3 and 10, 5 and 6; and the sum of the last two numbers is 11. Hence,

$$x^2 + 11x + 30 = (x + 5)(x + 6).$$

2. Resolve into factors $x^2 - 7x + 12$.

We are required to find two numbers whose product is 12 and whose algebraic sum is -7.

Since the product is $+12$, the two numbers are *both positive* or *both negative*; and since their sum is -7, they must both be negative.

Two negative numbers whose product is 12 are -12 and -1, -6 and -2, -4 and -3; and the sum of the last two numbers is -7. Hence,

$$x^2 - 7x + 12 = (x - 4)(x - 3).$$

3. Resolve into factors $x^2 + 2x - 24$.

We are required to find two numbers whose product is -24 and whose algebraic sum is 2.

Since the product is -24, one of the numbers is positive and the other negative; and since their sum is $+2$, the larger number is positive.

Two numbers whose product is -24, and the larger number positive, are 24 and -1, 12 and -2, 8 and -3, 6 and -4; and the sum of the last two numbers is $+2$. Hence,

$$x^2 + 2x - 24 = (x + 6)(x - 4).$$

4. Resolve into factors $x^2 - 3x - 18$.

Since the product is -18, one of the numbers is positive and the other negative; and since their sum is -3, the larger number is negative.

Two numbers whose product is -18, and the larger number negative, are -18 and 1, -9 and 2, -6 and 3; and the sum of the last two numbers is -3. Hence,

$$x^2 - 3x - 18 = (x - 6)(x + 3).$$

5. Resolve into factors $x^2 - 10xy + 9y^2$.

We are required to find two numbers whose product is $9y^2$ and whose algebraic sum is $-10y$.

Since the product is $+9y^2$, and the sum $-10y$, the last two terms must both be negative.

Two negative numbers whose product is $9y^2$ are $-9y$ and $-y$, $-3y$ and $-3y$; and the sum of the first two numbers is $-10y$. Hence,

$$x^2 - 10xy + 9y^2 = (x - 9y)(x - y).$$

131. From these examples it will be seen that the following statements are true:

1. If the *third* term of a given trinomial is *negative*, the *second* terms of its binomial factors have *unlike signs*.

2. If the *third* term of a given trinomial is *positive*, the *second* terms of its binomial factors have the *same sign*, and this sign is the *sign of the middle term*.

EXERCISE 43.

Resolve into factors:

1. $x^2 + 8x + 15$.
2. $x^2 - 8x + 15$.
3. $x^2 + 2x - 15$.
4. $x^2 - 3x - 10$.
5. $x^2 + 5ax + 6a^2$.
6. $x^2 - 5ax + 6a^2$.
7. $x^2 - 2x - 15$.
8. $x^2 + 5x + 6$.
9. $x^2 - 5x + 6$.
10. $x^2 + x - 6$.
11. $x^2 - x - 6$.
12. $x^2 + 6x + 5$. +1+5
13. $x^2 - 6x + 5$.
14. $x^2 + 4x - 5$.
15. $x^2 - 4x - 5$.
16. $x^2 + 9x + 18$.
17. $x^2 - 9x + 18$.
18. $x^2 + 3x - 18$.
19. $x^2 - 3x - 18$.
20. $x^2 + 9x + 8$.
21. $x^2 - 9x + 8$.
22. $x^2 + 7x - 8$.
23. $x^2 - 7x - 8$.
24. $x^2 + 7x + 10$.
25. $x^2 - 7x + 10$.
26. $x^2 + 3x - 10$.
27. $x^2 + ax - 6a^2$.
28. $x^2 - ax - 6a^2$.
29. $x^2 + 5xy + 4y^2$.
30. $x^2 - 3xy - 4y^2$.
31. $x^2 - 5xy + 4y^2$.
32. $x^2 + 3xy - 4y^2$.
33. $x^2 + 3xy + 2y^2$.
34. $a^2 - 7ab + 10b^2$.
35. $a^2x^2 - 3ax - 54$.
36. $x^2 - 7x - 44$.
37. $x^2 + x - 132$.
38. $x^2 - 15x + 50$.
39. $a^2 - 23a + 120$.
40. $a^2 + 17a - 390$.
41. $c^2 + 25c - 150$.
42. $c^2 - 58c + 57$.
43. $a^4 - 11a^2b^3 + 30b^6$.
44. $z^2 + 9zy + 20y^2$.
45. $x^2y^2 + 19xyz + 48z^2$.
46. $a^2b^2 - 13abc + 22c^2$.

132. When a trinomial has the form ax^2+bx+c.

1. Find the factors of $8x^2-22x-21$.

Multiply by 8, the coefficient of x^2, and write the result in the following form:

$$(8x)^2-22\times 8x-168.$$

Put z for $8x$, and we have

$$z^2-22z-168.$$

Resolve this expression into two binomial factors (§ 130)

$$(z-28)(z+6).$$

Since we have multiplied by 8, and put z for $8x$, we must reverse this process. Hence, put $8x$ for z and divide by 8, and we have

$$\frac{(8x-28)(8x+6)}{8}.$$

As 4 is a factor of $(8x-28)$, and 2 is a factor of $(8x+6)$, we divide by 8 by dividing the first factor by 4 and the second factor by 2, thus

$$\frac{(8x-28)(8x+6)}{4\times 2}$$
$$=(2x-7)(4x+3).$$

2. Find the factors of $24x^2-70xy-75y^2$.

Multiply by 24, $(24x)^2-70y\times 24x-1800y^2$.
Put z for $24x$, $z^2-70yz-1800y^2$.
Resolve into factors, $(z-90y)(z+20y)$. (§ 130)
Put $24x$ for z, $(24x-90y)(24x+20y)$.
Divide by 6×4, $(4x-15y)(6x+5y)$.

3. Find the factors of $12x^2-23xy+10y^2$.

Multiply by 12, $(12x)^2-23y\times 12x+120y^2$.
Put z for $12x$, $z^2-23yz+120y^2$.
Resolve into factors, $(z-15y)(z-8y)$. (§ 130)
Put $12x$ for z, $(12x-15y)(12x-8y)$.
Divide by 3×4, $(4x-5y)(3x-2y)$.

EXERCISE 44.

Resolve into factors:

1. $2x^2 + 5x + 3$.
2. $3x^2 - x - 2$.
3. $5x^2 - 8x + 3$.
4. $6x^2 + 7x + 2$.
5. $6x^2 - x - 2$.
6. $15x^2 + 14x - 8$.
7. $8x^2 - 10x + 3$.
8. $18x^2 + 9x - 2$.
9. $12x^2 - 5x - 2$.
10. $12x^2 - 7x + 1$.
11. $12x^2 - x - 1$.
12. $3x^2 - 2x - 5$.
13. $3x^2 + 4x - 4$.
14. $6x^2 + 5x - 4$.
15. $4x^2 + 13x + 3$.
16. $4x^2 + 11x - 3$.
17. $4x^2 - 4x - 3$.
18. $4x^2 + 8x + 3$.
19. $6a^2x^2 + ax - 1$.
20. $6x^2 + 17x + 12$.
21. $12x^2 - 13x - 14$.
22. $10a^2 - 23a - 5$.
23. $8c^2 + 53c - 21$.
24. $8z^2 - 37z - 15$.
25. $2x^2 + 5xy + 2y^2$.
26. $6x^2 - 7bx - 3b^2$.
27. $8a^2 + 14ab - 15b^2$.
28. $6a^2 - 19ac + 10c^2$.
29. $8x^2 + 34xy + 21y^2$.
30. $8x^2 - 22xy - 21y^2$.
31. $6x^2 + 19xy - 7y^2$.
32. $11a^2 - 23ab + 2b^2$.
33. $2c^2 - 13cd + 6d^2$.
34. $6y^2 + 7yz - 3z^2$.
35. $15x^2 - 26xy + 8y^2$.
36. $9x^2 + 6xy - 8y^2$.
37. $6x^2 - xy - 35y^2$.
38. $10x^2 - 21xy - 10y^2$.
39. $14x^2 - 55xy + 21y^2$.
40. $6x^2 - 23xy + 20y^2$.
41. $6x^2 + 35xy - 6y^2$.
42. $24x^2 - 14xy - 5y^2$.
43. $24x^2 - 38xy + 15y^2$.
44. $24x^2 - 2xy - 15y^2$.
45. $36x^2 - 19xy - 6y^2$.
46. $15x^2 + 19xy + 6y^2$.
47. $12x^2 + 31xy - 15y^2$.
48. $5y^2 + 13y - 6$.

133. When a binomial is the sum or difference of two cubes.

Since $$\frac{a^3+b^3}{a+b}=a^2-ab+b^2; \qquad (\S\ 117)$$

and $$\frac{a^3-b^3}{a-b}=a^2+ab+b^2; \qquad (\S\ 117)$$

$$\therefore a^3+b^3=(a+b)(a^2-ab+b^2) \qquad (1)$$

and $$a^3-b^3=(a-b)(a^2+ab+b^2). \qquad (2)$$

Therefore, the sum of two perfect cubes is divisible by the sum of their cube roots, and the difference of two perfect cubes is divisible by the difference of their cube roots.

1. Resolve into factors $8a^3+27b^6$.

The cube root of $8a^3$ is $2a$, and of $27b^6$ is $3b^2$.

By putting $2a$ for a and $3b^2$ for b in (1), we have

$$(2a)^3+(3b^2)^3=(2a+3b^2)(4a^2-6ab^2+9b^4).$$

2. Resolve into factors $125x^3-1$.

The cube root of $125x^3$ is $5x$, and of 1 is 1.

By putting $5x$ for a and 1 for b in (2), we have

$$125x^3-1=(5x-1)(25x^2+5x+1).$$

3. Resolve into factors x^6+y^9.

The cube root of x^6 is x^2, and of y^9 is y^3.

By putting x^2 for a and y^3 for b in (1), we have

$$x^6+y^9=(x^2+y^3)(x^4-x^2y^3+y^6).$$

4. Resolve into factors $(x-y)^3+z^3$.

The cube root of $(x-y)^3$ is $x-y$, and of z^3 is z.

By putting $x-y$ for a and z for b in (1), we have

$$\begin{aligned}(x-y)^3+z^3&=[(x-y)+z][(x-y)^2-(x-y)z+z^2]\\&=(x-y+z)(x^2-2xy+y^2-xz+yz+z^2).\end{aligned}$$

EXERCISE 45.

Resolve into factors:

1. $a^3 + 8b^3$.
2. $a^3 - 27a^6$.
3. $a^3 + 64$.
4. $125a^3 + 1$.
5. $27x^3y^3 - 1$.
6. $a^3 + 27b^3$.
7. $x^3y^3 - 64$.
8. $64a^6 + 125b^3$.
9. $216a^6 - b^3$.
10. $64a^3 - 27b^3$.
11. $343 - x^3$.
12. $a^3b^3 + 343$.
13. $8a^3 - b^6$.
14. $216m^3 + n^6$.
15. $x^3y^3 - 512z^3$.
16. $729a^6 + 216c^6$.
17. $729y^3 - 64z^3$.
18. $512a^3 - 1$.
19. $(a+b)^3 - 1$.
20. $(a-b)^3 + 1$.
21. $1 - (a-b)^3$.
22. $27x^6 + 125y^3$.
23. $216 + 343y^6$.
24. $x^{15} - 125x^6$.
25. $8x^{18} + 27y^{12}$.
26. $a^{15} - 216b^9$.
27. $c^3d^3 - 343z^6$.
28. $x^3y^3z^6 + 8$.
29. $8x^3y^6z^9 - 1$.
30. $27x^9y^3z^6 + 1$.
31. $64x^3 - 125y^9$.
32. $125x^6 + 27y^{15}$.
33. $8x^3 - (x-y)^3$.
34. $8(x+y)^3 + z^3$.
35. $(a+b)^3 - (a-b)^3$.
36. $(x^2-3)^3 - z^3$.
37. $b^3 + (a-c)^3$.
38. $(x-1)^3 - (z-1)^3$.
39. $(a-3b)^3 - c^3$.
40. $(2-3a)^3 + b^3$.
41. $(z+5)^3 - a^3$.
42. $(y-z)^3 + (y+z)^3$.
43. $(2x+y)^3 - (x-y)^3$.
44. $1 - (7a-5b)^3$.
45. $(3x+y)^3 - 8z^3$.
46. $(2a-b)^3 - c^3$.
47. $x^3 - (y-z)^3$.
48. $125 - (x-2y)^3$.
49. $(2x+y)^3 - 27z^3$.
50. $(a-3)^3 - z^3$.
51. $(7a-b)^3 + c^3$.
52. $(3c-2y)^3 - 8z^3$.

Theory of Divisors.

134. Theorem. *The expression $x - y$ is an exact divisor of $x^n - y^n$ when n is any positive integer.*

Since $-x^{n-1}y + x^{n-1}y = 0$, (§ 34)

$$x^n - y^n = x^n - x^{n-1}y + x^{n-1}y - y^n.$$

Taking out x^{n-1} from the first two terms of the right side, and y from the last two terms, we have

$$x^n - y^n = x^{n-1}(x - y) + y(x^{n-1} - y^{n-1}).$$

Now $x - y$ is an exact divisor of the right side, if it is an exact divisor of $x^{n-1} - y^{n-1}$; and if $x - y$ is an exact divisor of the right side, it is an exact divisor of the left side; that is, $x - y$ is an exact divisor of $x^n - y^n$ *if* it is an exact divisor of $x^{n-1} - y^{n-1}$.

But $x - y$ *is* an exact divisor of $x^3 - y^3$ (§ 117), therefore it is an exact divisor of $x^4 - y^4$; and since it is an exact divisor of $x^4 - y^4$, it is an exact divisor of $x^5 - y^5$; and so on, indefinitely.

The method employed in proving this Theorem is called **Proof by Mathematical Induction.**

135. The Factor Theorem. *If a rational and integral expression in x vanishes, that is, becomes equal to 0, when r is put for x, then $x - r$ is an exact divisor of the expression.*

Given $ax^n + bx^{n-1} + \cdots\cdots + hx + k$ (1)

By supposition, $ar^n + br^{n-1} + \cdots\cdots + hr + k = 0.$ (2)

By subtracting (2) from (1), the given expression assumes the form

$$a(x^n - r^n) + b(x^{n-1} - r^{n-1}) + \cdots\cdots + h(x - r).$$

But $x - r$ is an exact divisor of $x^n - r^n$, $x^{n-1} - r^{n-1}$, and so on.

Therefore, $x - r$ is an exact divisor of the given expression.

Note. If $x - r$ is an exact divisor of the given expression, r is an exact divisor of k; for k, the last term of the dividend, is equal to r, the last term of the divisor, multiplied by the last term of the quotient. Therefore, in searching for numerical values of x that will make the given expression vanish, only exact divisors of the last term of the expression need be tried.

1. Resolve into factors $x^3 + 3x^2 - 13x - 15$.

The exact divisors of 15 are 1, -1, 3, -3, 5, -5, 15, -15.

If we put 1 for x in $x^3 + 3x^2 - 13x - 15$, the expression does not vanish. If we put -1 for x, the expression vanishes.

Therefore, $x - (-1)$, that is, $x + 1$, is a factor.

Divide the expression by $x + 1$, and we have

$$\begin{aligned} x^3 + 3x^2 - 13x - 15 &= (x + 1)(x^2 + 2x - 15) \\ &= (x + 1)(x - 3)(x + 5). \end{aligned}$$

NOTE. An expression can sometimes be resolved into three or more factors.

2. Resolve into factors $x^3 - 26x - 5$.

By trial we find that the only exact divisor of -5 that makes the expression vanish is -5.

Therefore, divide by $x + 5$, and we have

$$x^3 - 26x - 5 = (x + 5)(x^2 - 5x - 1).$$

As neither $+1$ nor -1, the exact divisors of -1, will make $x^2 - 5x - 1$ vanish, this expression cannot be resolved into factors.

EXERCISE 46.

Resolve into factors:

1. $x^3 - 10x - 3$.
2. $x^3 - 26x + 5$.
3. $x^3 - 15x - 4$.
4. $x^3 - 8x + 3$.
5. $x^3 + 2x^2 + 9$.
6. $x^3 - 3x + 2$.
7. $x^3 - 12x + 16$.
8. $x^3 + 4x^2 - 5$.
9. $4x^3 - 7x + 3$.
10. $x^3 - 3x^2 + 4x - 2$.
11. $x^3 + 9x^2 + 16x + 4$.
12. $x^3 + 2x^2 - 34x - 5$.
13. $4x^3 - 12x^2 + 9x - 1$.
14. $x^3 - 2x^2 - 23x + 60$.
15. $6x^3 - 23x^2 + 16x - 3$.
16. $x^3 - 10x^2 + 33x - 36$.
17. $x^3 + 7x^2 + 12x + 4$.
18. $x^3 + 5x^2 + 7x + 2$.

136. A compound expression involving x and y is divisible by $x - y$ if the expression vanishes when $+y$ is put for x; and is divisible by $x + y$ if the expression vanishes when $-y$ is put for x.

If n is a positive integer, prove by the Factor Theorem:

1. $x^n + y^n$ is never divisible by $x - y$.

   Put y for x in $x^n + y^n$; then $x^n + y^n = y^n + y^n = 2y^n$.
   As $2y^n$ is not zero, $x^n + y^n$ is not divisible by $x - y$.

2. $x^n - y^n$ is divisible by $x + y$, **if n is even.**

   Put $-y$ for x in $x^n - y^n$, then $x^n - y^n = (-y)^n - y^n$.
   If n is even, $(-y)^n = y^n$, and $(-y)^n - y^n = y^n - y^n$.
   As $y^n - y^n = 0$, $x^n - y^n$ is divisible by $x + y$, **if n is even.**

3. $x^n + y^n$ is divisible by $x + y$, **if n is odd.**

   Put $-y$ for x in $x^n + y^n$, then $x^n + y^n = (-y)^n + y^n$.
   If n is odd, $(-y)^n = -y^n$, and $(-y)^n + y^n = -y^n + y^n$.
   As $-y^n + y^n = 0$, $x^n + y^n$ is divisible by $x + y$, **if n is odd.**

From § 134 and these three cases, we have

1. For all positive integral values of n,

$$x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots\cdots + y^{n-1}).$$

2. For all positive *even* integral values of n,

$$x^n - y^n = (x + y)(x^{n-1} - x^{n-2}y + x^{n-3}y^2 - \cdots\cdots - y^{n-1}).$$

3. For all positive *odd* integral values of n,

$$x^n + y^n = (x + y)(x^{n-1} - x^{n-2}y + x^{n-3}y^2 - \cdots\cdots + y^{n-1}).$$

4. $x^n + y^n$ is never divisible by $x - y$; and is not divisible by $x + y$, *if n is even.*

NOTE. In applying the preceding rules for resolving an expression into factors, *if the terms have a common monomial factor*, this factor should be removed first.

When an expression can be expressed as the *difference of two perfect squares*, the method of § 126 should be employed in preference to any other.

EXERCISE 47. — REVIEW.

Resolve into factors:

1. $a^3 - 9a$.
2. $4x^4 - x^2$.
3. $x^{12} + y^{12}$.
4. $x^4 - y^4$.
5. $x^3 + y^3$.
6. $x^6 - y^6$.
7. $36x^2 - 49y^2$.
8. $x^6 + y^6$.
9. $x^2 - 14x + 49$.
10. $x^2 - (a - b)^2$.
11. $a^2 - (m + n)^2$.
12. $x^2 - 11x + 18$.
13. $x^2 + 4x - 45$.
14. $x^2 + 13x + 36$.
15. $x^2 - 13x - 48$.
16. $x^2 + 9x - 36$.
17. $x^2 + x - 110$.
18. $2x^2 + 3xy - 2y^2$.
19. $z^2 - 6z - 40$.
20. $x^4 + x^2y^2 + y^4$.
21. $x^2 - 7x - 60$.
22. $2a^2 - 7a + 6$.
23. $x^2y^2 - 4xy^4 - 3x^2y^3$.
24. $x^3 + x^2 + x + 1$.
25. $3x^2 + x - 2$.
26. $x^2 - 24x + 95$.
27. $9a^2 + 12a + 4$.
28. $a^2 - b^2 - c^2 + 2bc$.
29. $x^2 - 2y + 2x - xy$.
30. $3x^3 + 2x^2 - 9x - 6$.
31. $(x - y)^2 - b^2$.
32. $m^2 - 2mn + n^2 - 1$.
33. $x^2 + 2ax + 3bx + 6ab$.
34. $x^2 + m^2 - n^2 - 2mx$.
35. $9x^4 + 21x^2y^2 + 25y^4$.
36. $x^2 - 4 + y^2 + 2xy$.
37. $6x^2 - x - 77$.
38. $x^2 - 8xy - 65y^2$.
39. $x^4 - 7x^2 + 1$.
40. $1 - a^2 - b^2 + 2ab$.
41. $3x^4 - 6x^3 + 9x^2$.
42. $x^3 - 5x^2 - 2x + 10$.
43. $x^2 + ax - bx - ab$.
44. $2x^2 - 3xy + 4ax - 6ay$.

45. $x^5 + y^5$.

46. $32x^5 - c^5$.

47. $a^6 + 64y^6$.

48. $729 - x^6$.

49. $x^{12} - y^{12}$.

50. $(a + b)^4 - 1$.

51. $16x^4 - 81$.

52. $x^4 + x^2 + 1$.

53. $27x^3 - 64a^3$.

54. $x^9 + y^9$.

55. $x^9 - y^9$.

56. $a^8 - 256$.

57. $1 - (x - y)^3$.

58. $x^6 - 216$.

59. $a^2 - 4a - 77$.

60. $a^3 + b^3 + a + b$.

61. $a^3 - b^3 + a - b$.

62. $a^2 - b^2 + a - b$.

63. $ax^4 + bx^3 - ax - b$.

64. $(x - y)^2 - 2y(x - y)$.

65. $1 - 10xy + 25x^2y^2$.

66. $a^2 - b^2 + 2bc - c^2$.

67. $x^2 + 4y^2 - z^2 - 4xy$.

68. $a^2 - 4b^2 - 9c^2 + 12bc$.

69. $4x^2 + 9y^2 - z^2 - 12xy$.

70. $(a + b)^2 - (c - d)^2$.

71. $a^2 + a + 3b - 9b^2$.

72. $5c^4 - 15c^3 - 90c^2$.

73. $a^2x - c^2x + a^2y - c^2y$.

74. $x^4 + 16a^2x^2 + 256a^4$.

75. $(x + y)^3 + (2x - y)^3$.

76. $a^2 + b^2 - c^2 + 2ab$.

77. $y^2 - a^2 - c^2 - 2ac$.

78. $(x + 5a)^2 - 25a^2$.

79. $2xy - x^2 - y^2 + z^2$.

80. $4a^4 - 9a^2 + 6a - 1$.

81. $a^2 - 2ab + b^2 + 12xy - 4x^2 - 9y^2$.

82. $2x^2 - 4xy + 2y^2 + 2ax - 2ay$.

83. $(a + b)^2 - 1 - ab(a + b + 1)$.

84. $x^3 - x^2 + 3x + 5$.

85. $x^2 - y^2 - z^2 - 2yz + x + y + z$.

86. $x^2 + y^2 + z^2 - 2xy - 2xz + 2yz$.

87. $4a^2b^2 - (a^2 + b^2 - c^2)^2$.

88. $x^2 + 2x - 3.$

89. $x^2 + 3x - 40.$

90. $x^2 - 9x - 10.$

91. $x^4 + 8x^2 - 9.$

92. $x^4 - 2x^3 - 24x^2.$

93. $x^4 - 14x^2 - 51.$

94. $x^2 - 10ax + 16a^2.$

95. $x^4 - 15x^2y^2 + 9y^4.$

96. $1 - 9x - 10x^2.$

97. $x^2 + 7xy + 12y^2.$

98. $x^2 + xy - 2y^2.$

99. $x^2 + 3xy - 4y^2.$

100. $x^4 - (y + z)^4.$

101. $x^3 - y^3 - 3xy(x - y).$

102. $x^3 - 2x^2 + 2 - x.$

103. $x^3 - 8 - 6x + 3x^2.$

104. $3x^2y^2 + 9xy^3 - 12y^4.$

105. $a^4 - a^3b + ab^3 - b^4.$

106. $a^2 - 4c^2 + a - 2c.$

107. $4a^2 + 9b^2 - c^2 + 12ab.$

108. $15x^2 - 5ax + 3bx - ab.$

109. $3a^4 + 15a^3b - 24a^2b^2.$

110. $6a^3 - 30a^2b + 36ab^2.$

111. $25a^2 - 4x^2 + 4x - 10a.$

112. $x^4y - x^2y^3 - x^3y^2 + xy^4.$

113. $9a^2 - 4b^2 + 3a + 2b.$

114. $x^3 - y^3 - (x^2 - y^2) - (x - y)^2.$

115. $(x - y)^2 - 1 - 2(x - y - 1).$

116. $a^3 - 2a^2c + a^2 - 4a + 8c - 4.$

117. $a^2 - b^2 - c^2 + 2bc + a + b - c.$

118. $x^3z^2 - 8y^3z^2 - 4x^3n^2 + 32y^3n^2.$

119. $5ac + 3bc + c + 5ab + 3b^2 + b.$

120. $2ab - 2bc - ax + cx + 2b^2 - bx.$

121. $x^4 - 2abx^2 - a^4 - a^2b^2 - b^4.$

CHAPTER VIII.

COMMON FACTORS AND MULTIPLES.

Highest Common Factor.

137. Common Factors. A **common factor** of two or more *integral numbers* in Arithmetic is an integral number that divides each of them without a remainder.

138. A **common factor** of two or more *integral and rational expressions* in Algebra is an integral and rational expression that divides each of them without a remainder.

Thus, $5a$ is a common factor of $20a$ and $25a$.

139. Two *numbers* in Arithmetic are said to be **prime** to each other when they have no common factor except 1.

140. Two *expressions* in Algebra are said to be **prime** to each other when they have no common factor except 1.

141. The **greatest common factor** of two or more *integral numbers* in Arithmetic is the greatest number that will divide each of them without a remainder.

142. The **highest common factor** of two or more *integral and rational expressions* in Algebra is an integral and rational expression of *highest degree* that will divide each of them without a remainder.

Thus, $3a^2$ is the highest common factor of $3a^2$, $6a^3$, and $12a^4$; $5x^2y^2$ is the highest common factor of $10x^3y^2$ and $15x^2y^2$.

For brevity, we use H. C. F. for highest common factor.

1. Find the H. C. F. of $42\,a^3b^2$ and $30\,a^2b^4$.

$$42\,a^3b^2 = 2 \times 3 \times 7 \times aaa \times bb;$$
$$30\,a^2b^4 = 2 \times 3 \times 5 \times aa \times bbbb.$$
$$\therefore \text{ the H. C. F.} = 2 \times 3 \times aa \times bb = 6\,a^2b^2.$$

2. Find the H. C. F. of $x^2 - 9\,y^2$ and $x^2 + 6\,xy + 9\,y^2$.

$$x^2 - 9\,y^2 = (x + 3\,y)(x - 3\,y);$$
$$x^2 + 6\,xy + 9\,y^2 = (x + 3\,y)(x + 3\,y).$$
$$\therefore \text{ the H. C. F.} = x + 3\,y.$$

3. Find the H. C. F. of

$$4\,x^2 - 4\,x - 80;\ 2\,x^2 - 18\,x + 40;\ 2\,x^2 - 24\,x + 70.$$

$$4\,x^2 - 4\,x - 80 = 4\,(x^2 - x - 20)$$
$$= 4\,(x - 5)(x + 4);$$
$$2\,x^2 - 18\,x + 40 = 2\,(x^2 - 9\,x + 20)$$
$$= 2\,(x - 5)(x - 4);$$
$$2\,x^2 - 24\,x + 70 = 2\,(x^2 - 12\,x + 35)$$
$$= 2\,(x - 5)(x - 7).$$
$$\therefore \text{ the H.C.F.} = 2\,(x - 5).$$ Therefore,

143. To Find the H. C. F. of Two or More Expressions,

Resolve each expression into its prime factors.

The product of all the common factors, each factor being taken the least number of times it occurs in any of the given expressions, is the highest common factor required.

Note. The *highest common factor* in Algebra corresponds to the *greatest common measure*, or *greatest common divisor*, in Arithmetic. We cannot apply the terms *greatest* and *least* to algebraic expressions in which particular values have not been given to the letters contained in the expressions. Thus a is *greater* than a^2, if a stands for $\frac{1}{4}$; but a is of *lower degree* than a^2.

EXERCISE 48.

Find the H.C.F. of:

1. $120a^2$ and $168a^3$.
2. $36x^3$ and $27x^4$.
3. $42a^2x^3$ and $60a^3x^2$.
4. $36a^3x^2$ and $28x^3y$.
5. $48a^2b^3c$ and $60a^3c^3$.
6. $8(a+b)^2$ and $6(a+b)^3$.
7. $12a(x+y)^2$ and $4b(x+y)^3$.
8. $(x-1)^2(x+2)^2$ and $(x-3)(x+2)^3$.
9. $24a^2b^3(a+b)$ and $42a^3b(a+b)^2$.
10. $x^2(x-3)^2$ and x^2-3x.
11. x^2-16 and x^2+ [redacted]
12. x^2-4x and x^2-6x+8.
13. $x^2-7x+12$ and x^2-16.
14. $9x^2-4y^2$ and $27x^3-8y^3$.
15. x^2-7x-8 and x^2+5x+4.
16. $x^2+3xy-10y^2$ and $x^2-2xy-35y^2$.
17. $x^4-2x^3-24x^2$ and $6x^5-6x^4-180x^3$.
18. x^3-3x^2y and x^3-27y^3.
19. $1+64x^3$ and $1-4x+16x^2$.
20. x^4-81 and x^4+8x^2-9.
21. x^2+2x-3 and $x^2+7x+12$.
22. x^2-6x+5 and $x^2+3x-40$.
23. $3a^4+15a^3b-72a^2b^2$ and $6a^3-30a^2b+36ab^2$.
24. $6x^2y-12xy^2+6y^3$ and $3x^2y^2+9xy^3-12y^4$.
25. $1-16c^4$ and $1+c^2-12c^4$.
26. $9ax^3-ax$; $9x^2-6x+1$; $27x^3-1$.
27. $x^2-3x-54$; x^2-x-42; $x^2-2x-48$.
28. $8x^3+27y^3$; $4x^2+12xy+9y^2$; $4x^2-9y^2$.
29. $x^3-x^2y-xy^2+y^3$; x^2-y^2; $x^2+2xy+y^2$.

X

Lowest Common Multiple.

144. Common Multiples. A **common multiple** of two or more *integral numbers* in Arithmetic is a number that is exactly divisible by each of the numbers.

A **common multiple** of two or more *integral and rational expressions* in Algebra is an integral and rational expression that is exactly divisible by each of the expressions.

145. The **least common multiple** of two or more *integral numbers* in Arithmetic is the *least* integral number that is exactly divisible by each of the given numbers.

The **lowest common multiple** of two or more *integral and rational expressions* in Algebra is an integral and rational expression of *lowest degree* that is exactly divisible by each of the given expressions.

We use L. C. M. for lowest common multiple.

1. Find the L. C. M. of $42a^3b^2$; $30a^2b^4$; $66ab^3$.

$$42a^3b^2 = 2 \times 3 \times 7 \times a^3 \times b^2;$$
$$30a^2b^4 = 2 \times 3 \times 5 \times a^2 \times b^4;$$
$$66ab^3 = 2 \times 3 \times 11 \times a \times b^3.$$

The L. C. M. must evidently contain each factor the greatest number of times that it occurs in any expression.

$$\therefore \text{the L.C.M.} = 2 \times 3 \times 7 \times 5 \times 11 \times a^3 \times b^4$$
$$= 2310a^3b^4.$$

2. Find the L. C. M. of

$$4x^2 - 4x - 80 \text{ and } 2x^2 - 18x + 40.$$

$$4x^2 - 4x - 80 = 4(x^2 - x - 20) = 4(x-5)(x+4);$$
$$2x^2 - 18x + 40 = 2(x^2 - 9x + 20) = 2(x-5)(x-4).$$
$$\therefore \text{the L.C.M.} = 4(x-5)(x+4)(x-4). \quad \text{Hence,}$$

146. To Find the L. C. M. of Two or More Expressions,

Resolve each expression into its prime factors.

The product of all the different factors, each factor being taken the greatest number of times it occurs in any of the given expressions, is the lowest common multiple required.

EXERCISE 49.

Find the L. C. M. of:

1. x^3y^2 and x^2y^3.
2. $5abc^2$ and $a^2b^2c^3$.
3. $4x^3y$ and $12xy^3$.
4. $5a^3b^3$ and $10a^2b^5$.
5. $21xy^3$ and $28x^3y^7$.
6. $10x^2z^3$ and $15x^2z^2$.
7. ab^3, $a^2b^2c^2$, and abc^3.
8. x^2y, xy^2z^2, and x^2y^3z.
9. x^2 and $x^2 + x$.
10. $x^2 - 1$ and $x^2 + x$.
11. $a^2 + ab$ and $ab + b^2$.
12. $a^2 + 2a$ and $(a + 2)^2$.
13. $x^2 + 4x + 4$ and $x^2 + 5x + 6$.
14. $x^2 + x - 20$ and $x^2 - x - 30$.
15. $y^2 + y - 42$ and $y^2 - 11y + 30$.
16. $z^2 - 10z + 24$ and $z^2 + z - 20$.
17. $(a + b)^2$; $(a - b)^2$; $a^2 - b^2$.
18. $(a + 2c)^2$; $(a - 2c)^2$; $a^2 - 4c^2$.
19. $4xy(x + y)^2$; $2x^2(x^2 - y^2)$; $x^3(x + y)$.
20. $x^2 + 7x + 12$; $x^2 + 6x + 8$; $x^2 + 5x + 6$.
21. $1 - y^2$; $1 - y^3$; $1 + y$.
22. $x^2 + 2xy + y^2$; $x^2 - y^2$; $x^2 - 2xy + y^2$.
23. $x^3 - 27$; $x^2 + 5x$; $x^2 + 2x - 15$.
24. $y^2 - 1$; $y^3 + y^2 + y + 1$; $y^3 - y^2 + y - 1$.
25. $(x + y)^2 - z^2$; $(x + y + z)^2$; $x + y - z$.
26. $x^2 - (a + b)x + ab$; $x^2 - (a + c)x + ac$.
27. $x^2 + 3xy + 2y^2$; $x^2 + 5xy + 4y^2$; $x^2 - 6xy - 7y^2$.
28. $x^2 - 7xy + 12y^2$; $x^2 - 6xy + 8y^2$; $x^2 - 5xy + 6y^2$.

147. The chief difficulty in finding the H. C. F. and the L. C. M. of two or more algebraic expressions consists in resolving the expressions into factors.

When two numbers in Arithmetic cannot readily be resolved into their prime factors, we divide the greater number by the smaller; then the divisor by the remainder; and so on until there is no remainder. The last divisor is the greatest common factor.

Likewise in Algebra when two given expressions cannot readily be resolved into their factors, we arrange the two given expressions in descending powers of a common letter, and divide the expression which is of higher degree in the common letter by the other expression. After the first division, we take the remainder for a new divisor and the divisor for a new dividend, and so proceed until there is no remainder. The last divisor is the highest common factor.

NOTE. If the two expressions are of the same degree in the common letter, either expression may be taken for the divisor.

Find the H. C. F. of $2x^2 + x - 3$ and $4x^3 + 8x^2 - x - 6$.

$$\begin{array}{llll}
2x^2 + x - 3) & 4x^3 + 8x^2 - \quad x - 6\,(2x + 3 & & \\
 & \underline{4x^3 + 2x^2 - 6x} & & \\
 & \qquad\quad 6x^2 + 5x - 6 & & \\
 & \qquad\quad \underline{6x^2 + 3x - 9} & & \\
 & & 2x + 3) & 2x^2 + \quad x - 3\,(x - 1 \\
 & & & \underline{2x^2 + 3x} \\
 & & & \qquad - 2x - 3 \\
 & & & \qquad \underline{- 2x - 3}
\end{array}$$

$\therefore$ the H. C. F. $= 2x + 3$.

NOTE. Each division is continued until the first term of the remainder is of lower degree than the first term of the divisor.

148. This method is of use only to obtain the **compound factor** of the H. C. F. **Monomial factors** of the given expressions *must first be separated from them*, and the H. C. F. of these monomial factors must be reserved to be multiplied into the compound factor obtained. Also, at any stage of the operation a monomial factor of either expression may be removed without affecting the compound factor sought.

1. Find the H. C. F. of

$$12x^4+30x^3-72x^2 \text{ and } 32x^3+84x^2-176x.$$

$$12x^4+30x^3-72x^2=6x^2(2x^2+5x-12).$$
$$32x^3+84x^2-176x=4x(8x^2+21x-44).$$
$6x^2$ and $4x$ have $2x$ *common.*

$$\begin{array}{lll}
2x^2+5x-12) & 8x^2+21x-44\,(4 & \\
 & \underline{8x^2+20x-48} & \\
 & \qquad\quad x+4) & 2x^2+5x-12\,(2x-3 \\
 & & \underline{2x^2+8x\qquad\quad} \\
 & & \qquad -3x-12 \\
 & & \qquad \underline{-3x-12}
\end{array}$$

$\therefore$ the H. C. F. $=2x(x+4)$.

2. Find the H. C. F. of $4x^2-8x-5$ and $12x^2-4x-65$.

$$\begin{array}{ll}
4x^2-8x-5) & 12x^2-4x-65\,(3 \\
 & \underline{12x^2-24x-15} \\
 & \qquad\quad 20x-50
\end{array}$$

The first division ends here, for $20x$ is of lower degree than $4x^2$. We take out the *simple factor* 10 from $20x-50$, for 10 is not a factor of the given expressions, and its rejection can in no way affect the *compound factor sought*, and proceed with $2x-5$ for a divisor.

$$\begin{array}{ll}
2x-5) & 4x^2-8x-5\,(2x+1 \\
 & \underline{4x^2-10x\qquad\ } \\
 & \qquad\quad 2x-5 \\
 & \qquad\quad \underline{2x-5}
\end{array}$$

$\therefore$ the H. C. F. $=2x-5$.

3. Find the H.C.F. of

$$21x^3-4x^2-15x-2 \text{ and } 21x^3-32x^2-54x-7.$$

$$\begin{array}{rl}
21x^3-4x^2-15x-2) & 21x^3-32x^2-54x-7\,(1 \\
 & \underline{21x^3-4x^2-15x-2} \\
 & -28x^2-39x-5
\end{array}$$

The difficulty here cannot be obviated by *removing* a simple factor from the remainder, for $-28x^2-39x-5$ has no simple factor. In this case, to avoid the inconvenience of fractions we *multiply* the expression $21x^3-4x^2-15x-2$ by the simple factor 4 to make its first term exactly divisible by $-28x^2$.

The *introduction* of such a factor can in no way affect the H.C.F. sought, for 4 is not a factor of either of the given expressions; and if we multiply only *one* of the expressions by 4 we do not introduce a *common* factor.

The *signs* of all the terms of the remainder may be changed; for if an expression A is divisible by $-F$, it is divisible by $+F$.

The process then is continued by changing the signs of all the terms of the remainder and multiplying the divisor by 4.

$$\begin{array}{lrl}
 & 28x^2+39x+5) & 84x^3-16x^2-60x-8\,(3x \\
 & & \underline{84x^3+117x^2+15x\phantom{-8}} \\
 & & -133x^2-75x-8 \\
\text{Multiply by } -4, & & \underline{-4\phantom{x^2-75x-8}} \\
 & & 532x^2+300x+32\,(19 \\
 & & \underline{532x^2+741x+95} \\
\text{Divide by } -63, & & -63)\underline{-441x-63} \\
 & & 7x+1
\end{array}$$

$$\begin{array}{rl}
7x+1) & 28x^2+39x+5\,(4x+5 \\
 & \underline{28x^2+4x} \\
 & 35x+5 \\
 & 35x+5
\end{array}$$

$\therefore$ the H.C.F. $=7x+1$.

4. Find the H.C.F. of

$$8x^2 + 2x - 3 \text{ and } 6x^3 + 5x^2 - 2.$$

$$\begin{array}{lrll}
 & & 6x^3 + 5x^2 - 2 & \\
\text{Multiply by 4,} & & 4 & \\ \cline{3-3}
 & 8x^2 + 2x - 3) & 24x^3 + 20x^2 - 8 & (3x + 7 \\
 & & 24x^3 + 6x^2 - 9x & \\ \cline{3-3}
 & & \quad 14x^2 + 9x - 8 & \\
\text{Multiply by 4,} & & \quad 4 & \\ \cline{3-3}
 & & \quad 56x^2 + 36x - 32 & \\
 & & \quad 56x^2 + 14x - 21 & \\ \cline{3-3}
\text{Divide by 11,} & & \qquad 11\underline{)\,22x - 11} &
\end{array}$$

$$\begin{array}{rl}
2x - 1) & 8x^2 + 2x - 3\,(4x + 3 \\
 & 8x^2 - 4x \\ \hline
 & \qquad 6x - 3 \\
 & \qquad 6x - 3 \\ \hline
\end{array}$$

$\therefore$ the H.C.F. $= 2x - 1$.

The following arrangement of the work will be found most convenient:

$$\begin{array}{r|r|l}
8x^2 + 2x - 3 & 6x^3 + 5x^2 - 2 & \\
\underline{8x^2 - 4x} & \underline{4 \qquad\qquad\qquad} & \\
6x - 3 & 24x^3 + 20x^2 - 8 & 3x \\
\underline{6x - 3} & \underline{24x^3 + 6x^2 - 9x} & \\
 & 14x^2 + 9x - 8 & \\
 & \underline{4 \qquad\qquad\quad} & \\
 & 56x^2 + 36x - 32 & +7 \\
 & \underline{56x^2 + 14x - 21} & \\
 & 11\underline{)\,22x - 11} & \\
 & 2x - 1 & 4x + 3
\end{array}$$

NOTE. From the nature of division, the successive remainders are expressions of lower and lower degree. Hence, unless at some step the division leaves no remainder, we shall at last have a remainder that does not contain the common letter. In this case the given expressions have no H.C.F. that contains the common letter.

149. In the examples worked out we have *assumed* that the divisor which is contained in the corresponding dividend without a remainder is the H. C. F. required.

The *proof* may be given as follows:

Let A and B stand for two expressions which have no monomial factors, and which are arranged according to the descending powers of a common letter, the degree of B being not higher than that of A in the common letter.

Let A be divided by B, and let Q stand for the quotient, and R for the remainder. Then, since the dividend is equal to the product of the divisor and quotient plus the remainder, we have

$$A = BQ + R. \qquad (1)$$

Since the remainder is equal to the dividend minus the product of the divisor and quotient, we have

$$R = A - BQ. \qquad (2)$$

Now, a factor of each of the terms of an expression is a factor of the expression. Hence, any common factor of B and R is a factor of $BQ + R$, and by (1) a factor of A. That is, a common factor of B and R is also a common factor of A and B.

Also, any common factor of A and B is a factor of $A - BQ$, and by (2) a factor of R. That is, a common factor of A and B is also a common factor of B and R.

Therefore, the common factors of A and B are *the same* as the common factors of B and R; and consequently the H. C. F. of A and B is *the same* as the H. C. F. of B and R.

The proof for each succeeding step in the process is precisely the same; so that the H. C. F. of *any* divisor and the corresponding dividend is the H. C. F. required.

If at any step there is no remainder, the divisor is a factor of the corresponding dividend, and is therefore the H. C. F. of itself and the corresponding dividend. Hence, *this divisor* is the H. C. F. required.

150. The methods of resolving expressions into factors, given in the last chapter, often enable us to shorten the work of finding the H. C. F. required.

1. Find the H. C. F. of

$$x^4+3x^3+12x-16;\quad x^3-13x+12.$$

Both of these expressions vanish when 1 is put for x. Therefore, both are divisible by $x-1$, § 135.

The first quotient is $x^3+4x^2+4x+16=(x^2+4)(x+4)$.

The second quotient is $x^2+x-12=(x-3)(x+4)$.

Therefore, the H.C.F. is $(x-1)(x+4)$.

2. Find the H. C. F. of

$$2x^4+9x^3+14x+3;\quad 3x^4+14x^3+9x+2.$$

$$\begin{array}{r|l|l}
2x^4+9x^3+14x+3 & 3x^4+14x^3+9x+2 & \\
 & 2 & \\ \hline
 & 6x^4+28x^3+18x+4 & 3 \\
 & 6x^4+27x^3+42x+9 & \\ \hline
 & \qquad x^3-24x-5 &
\end{array}$$

The remainder, $x^3-24x-5$, vanishes when 5 is put for x.

The quotient of $x^3-24x-5$ divided by $x-5$ is x^2+5x+1.

Since 5 is not an exact divisor of 3, $x-5$ is not a factor of $2x^4+9x^3+14x+3$; but x^2+5x+1 is found by trial to be a factor, and is, therefore, the H. C. F. required.

3. Find the H. C. F. of

$$28x^2+39x+5;\quad 84x^3-16x^2-60x-8.$$

By § 132, the factors of $28x^2+39x+5$ are $7x+1$ and $4x+5$.

The factor $7x+1$ is the H. C. F. required.

4. Find the H. C. F. of

$$2x^4-6x^3-x^2+15x-10;\quad 4x^4+6x^3-4x^2-15x-15.$$

$$\begin{array}{r|l|l}
2x^4-6x^3-x^2+15x-10 & 4x^4+6x^3-4x^2-15x-15 & 2 \\
 & 4x^4-12x^3-2x^2+30x-20 & \\ \hline
 & \qquad 18x^3-2x^2-45x+5 &
\end{array}$$

The remainder $=2x^2(9x-1)-5(9x-1)$

$=(2x^2-5)(9x-1)$.

The factor $2x^2-5$ is the H. C. F. required.

5. Find the H. C. F. and the L. C. M. of:

$$6x^3 - 11x^2y + 2y^3 \text{ and } 9x^3 - 22xy^2 - 8y^3.$$

$$\begin{array}{l|l|l}
6x^3 - 11x^2y \qquad + 2y^3 & 9x^3 - 22xy^2 - 8y^3 & \\
\underline{6x^3 - 8x^2y - 4xy^2} & \underline{2} & \\
-3x^2y + 4xy^2 + 2y^3 & 18x^3 - 44xy^2 - 16y^3 & 3 \\
\underline{-3x^2y + 4xy^2 + 2y^3} & \underline{18x^3 - 33x^2y + 6y^3} & \\
 & 11y \,) \, \underline{33x^2y - 44xy^2 - 22y^3} & \\
 & \qquad 3x^2 - 4xy - 2y^2 & 2x - y
\end{array}$$

$\therefore$ the H. C. F. $= 3x^2 - 4xy - 2y^2$.

To find the L. C. M., divide each of the expressions by the H. C. F.

$$(6x^3 - 11x^2y + 2y^3) \div (3x^2 - 4xy - 2y^2) = 2x - y.$$

$$(9x^3 - 22xy^2 - 8y^3) \div (3x^2 - 4xy - 2y^2) = 3x + 4y.$$

$\therefore$ the L. C. M. $= (2x - y)(3x + 4y)(3x^2 - 4xy - 2y^2)$.

EXERCISE 50.

Find the H. C. F. and the L. C. M. of:

1. $4x^2 + 3x - 10$; $4x^3 + 7x^2 - 3x - 15$.
2. $2x^3 - 6x^2 + 5x - 2$; $8x^3 - 23x^2 + 17x - 6$.
3. $6x^3 - 7ax^2 - 20a^2x$; $3x^2 + ax - 4a^2$.
4. $3x^3 - 13x^2 + 23x - 21$; $6x^3 + x^2 - 44x + 21$.
5. $c^4 - 2c^3 + c$; $2c^4 - 2c^3 - 2c - 2$.
6. $a^3 - 6a^2x + 12ax^2 - 8x^3$; $2a^2 - 8ax + 8x^2$.
7. $7x^3 - 2x^2 - 5$; $7x^3 + 12x^2 + 10x + 5$.
8. $x^4 - 13x^2 + 36$; $x^4 - x^3 - 7x^2 + x + 6$.
9. $2x^3 + 3x^2 - 7x - 10$; $4x^3 - 4x^2 - 9x + 5$.
10. $12x^3 - x^2 - 30x - 16$; $6x^3 - 2x^2 - 13x - 6$.
11. $6x^3 + x^2 - 5x - 2$; $6x^3 + 5x^2 - 3x - 2$.
12. $x^3 - 9x^2 + 26x - 24$; $x^3 - 12x^2 + 47x - 60$.

13. $4x^3 - 2x^2 - 16x - 91$; $12x^3 - 28x^2 - 37x - 42$.

14. $x^4 - 4x^3 + 10x^2 - 12x + 9$; $x^4 + 2x^2 + 9$.

15. $2x^3 - 3x^2 - 16x + 24$; $4x^5 + 2x^4 - 28x^3 - 16x^2 - 32x$.

16. $12x^3 + 4x^2 + 17x - 3$; $24x^3 - 52x^2 + 14x - 1$.

17. $2x^3 + 7ax^2 + 4a^2x - 3a^3$; $4x^3 + 9ax^2 - 2a^2x - a^3$.

18. $2x^3 - 9ax^2 + 9a^2x - 7a^3$; $4x^3 - 20ax^2 + 20a^2x - 16a^3$.

19. $2x^4 + 9x^3 + 14x + 3$; $3x^4 + 14x^3 + 9x + 2$.

20. $20x^3 + 2x^2 - 18x + 48$; $20x^4 - 17x^2 + 48x - 3$.

21. $2x^3 + x^2 - 12x + 9$; $2x^3 - 7x^2 + 12x - 9$.

22. $x^3 - 8x + 3$; $x^6 - 3x^5 + 21x - 8$.

23. $3x^3 - 3x^2y + xy^2 - y^3$; $4x^3 - x^2y - 3xy^2$.

24. $8x^4 - 6x^3 - x^2 + 15x - 25$; $4x^3 + 7x^2 - 3x - 15$.

25. $4x^3 - 4x^2 - 5x + 3$; $10x^2 - 19x + 6$.

26. $6x^4 - 13x^3 + 3x^2 + 2x$; $6x^4 - 10x^3 + 4x^2 - 6x + 4$.

27. $2x^4 - 3x^3 + 2x^2 - 2x - 3$; $4x^4 + 3x^2 + 4x - 3$.

28. $3x^4 - x^3 - 2x^2 + 2x - 8$; $6x^3 + 13x^2 + 3x + 20$.

29. $3x^5 + 2x^4 + x^2$; $3x^4 + 2x^3 - 3x^2 + 2x - 1$.

30. $3 - 2x + 5x^2 + 2x^3$; $12 - 17x + 2x^2 + 3x^3$.

31. $10x - 6x^2 - 11x^3 + 9x^4 - 6x^5$;
$60x + 4x^2 + 10x^3 + 10x^4 + 4x^5$.

32. $x^4 - x^3 - 14x^2 + x + 1$; $x^5 - 4x^4 - x^3 - 2x^2 + 8x + 2$.

33. $2a^4 - 2a^3 - 3a^2 - 2a$; $3a^4 - a^3 - 2a^2 - 16a$.

34. $6x^3 - 14ax^2 + 6a^2x - 4a^3$; $x^4 - ax^3 - a^2x^2 - a^3x - 2a^4$.

35. $4 - 2x - 8x^2 + 7x^3 - 9x^5$; $2 + 5x - 10x^2 - 7x^3 + 6x^4$.

36. $2a^4 + 3a^3x - 9a^2x^2$; $6a^4x - 3ax^4 - 17a^3x^2 + 14a^2x^3$.

37. $2a^5 - 4a^4 + 8a^3 - 12a^2 + 6a$;
$3a^6 - 3a^5 - 6a^4 + 9a^3 - 3a^2$.

151. *The product of the H. C. F. and the L. C. M. of two expressions is equal to the product of the given expressions.*

Let A and B stand for any two expressions; and let F stand for their H. C. F. and M for their L. C. M.

Let a and b be the quotients when A and B respectively are divided by F. Then

$$A = aF$$

and

$$B = bF.$$

Therefore,

$$AB = F \times abF. \qquad (1)$$

Since F stands for the H. C. F. of A and B, F contains *all the common factors* of A and B. Therefore, a and b have no common factor, and abF is the L. C. M. of A and B.

Put M for its equal, abF, in equation (1), and we have

$$AB = FM.$$

152. Since

$$FM = AB, \qquad (\S\ 151)$$

$$M = \frac{AB}{F} = \frac{A}{F} \times B = \frac{B}{F} \times A.$$ That is,

The lowest common multiple of two expressions may be found by dividing their product by their highest common factor, or by dividing **either** *of them by their highest common factor and multiplying the quotient by the other.*

153. The H. C. F. of three or more expressions is obtained by finding the H. C. F. of two of them; then the H. C. F. of this result and of the third expression; and so on.

For, if A, B, and C stand for three expressions,
and D for the highest common factor of A and B,
and E for the highest common factor of D and C,
then D contains every factor common to A and B,
and E contains every factor common to D and C;
that is, every factor common to A, B, and C.

154. The L. C. M. of three or more expressions is obtained by finding the L. C. M. of two of them; then the L. C. M. of this result and of the third expression; and so on.

For, if A, B, and C stand for three expressions,
and L for the lowest common multiple of A and B,
and M for the lowest common multiple of L and C,
then L is the expression of lowest degree that is exactly divisible by A and B,
and M is the expression of lowest degree that is exactly divisible by L and C. That is, M is the expression of lowest degree that is exactly divisible by A, B, and C.

Exercise 51.

Find the H. C. F. and the L. C. M. of:

1. $6x^2 + x - 2$; $2x^2 + 7x - 4$; $2x^2 - 7x + 3$.
2. $a^2 + 2ab + b^2$; $a^2 - b^2$; $a^3 + 2a^2b + 2ab^2 + b^3$.
3. $x^2 - 5ax + 4a^2$; $x^2 - 3ax + 2a^2$; $3x^2 - 10ax + 7a^2$.
4. $x^2 + x - 6$; $x^3 - 2x^2 - x + 2$; $x^3 + 3x^2 - 6x - 8$.
5. $x^3 - 6x^2 + 11x - 6$; $x^3 - 8x^2 + 19x - 12$; $x^3 - 9x^2 + 26x - 24$.
   −1 −6
6. $6x^2 + 7xy - 3y^2$; $3x^2 + 11xy - 4y^2$; $2x^2 + 11xy + 12y^2$.
7. $8 - 14a + 6a^2$; $4a + 4a^2 - 3a^3$; $4a^2 + 2a^3 - 6a^4$.
8. $6x^3 + 7x^2 - 3x$; $3x^2 + 14x - 5$; $6x^2 + 39x + 45$.
9. $27x^3 - a^3$; $6x^2 + ax - a^2$; $15x^2 - 5ax + 3bx - ab$.
10. $x^3 - 1$; $2x^2 - x - 1$; $3x^2 - x - 2$.
11. $6x^2 - x - 2$; $21x^2 - 17x + 2$; $14x^2 + 5x - 1$.
12. $12x^2 + 2x - 4$; $12x^2 - 42x - 24$; $12x^2 - 28x - 24$.
13. $2x^2 + 3x - 5$; $3x^2 - x - 2$; $2x^2 + x - 3$.
14. $x^3 + 7x^2 + 5x - 1$; $x^2 + 3x - 3x^3 - 1$; $3x^3 + 5x^2 + x - 1$.

CHAPTER IX.

FRACTIONS.

Definitions.

155. An **algebraic fraction** is the indicated quotient of two expressions written in the form $\frac{a}{b}$.

156. The dividend a is called the **numerator**; the divisor b is called the **denominator**; the numerator and denominator are called the **terms** of the fraction.

Fundamental Principle of Fractions.

Let $$\frac{a}{b} = x. \tag{1}$$

Multiply by b, $$a = bx.$$

Multiply by c, $$ac = bcx.$$

Divide by bc, $$\frac{ac}{bc} = x. \tag{2}$$

From equations (1) and (2), $\frac{a}{b} = \frac{ac}{bc}$.

Now $\frac{ac}{bc}$ is obtained by multiplying both terms of $\frac{a}{b}$ by c; and $\frac{a}{b}$ is obtained by dividing both terms of $\frac{ac}{bc}$ by c. Hence,

157. *If the numerator and denominator of a fraction are both multiplied by the same number, or both divided by the same number, the value of the fraction is not altered.*

Reduction of Fractions to their Lowest Terms.

158. A fraction is in its *lowest terms* when the numerator and denominator have no common factor. Hence,

To Reduce a Fraction to Lowest Terms,

Resolve the numerator and denominator into their prime factors, and cancel all the common factors; or, *divide the numerator and denominator by their highest common factor.*

Reduce the following fractions to their lowest terms:

1. $\dfrac{38\,a^2b^3c^4}{57\,a^3bc^2} = \dfrac{2 \times 19\,a^2b^3c^4}{3 \times 19\,a^3bc^2} = \dfrac{2\,b^2c^2}{3\,a}.$

2. $\dfrac{a^3 - x^3}{a^2 - x^2} = \dfrac{(a - x)(a^2 + ax + x^2)}{(a - x)(a + x)} = \dfrac{a^2 + ax + x^2}{a + x}.$

3. $\dfrac{a^2 + 7\,a + 10}{a^2 + 5\,a + 6} = \dfrac{(a + 5)(a + 2)}{(a + 3)(a + 2)} = \dfrac{a + 5}{a + 3}.$

4. $\dfrac{x^3 - 4\,x^2 + 4\,x - 1}{x^3 - 2\,x^2 + 4\,x - 3}.$

We find by the Factor Theorem the H.C.F. of the numerator and denominator to be $x - 1$.

The numerator divided by $x - 1 \quad = x^2 - 3\,x + 1.$
The denominator divided by $x - 1 = x^2 - x + 3.$

Therefore, $\dfrac{x^3 - 4\,x^2 + 4\,x - 1}{x^3 - 2\,x^2 + 4\,x - 3} = \dfrac{x^2 - 3\,x + 1}{x^2 - x + 3}.$

Exercise 52.

Reduce to lowest terms:

1. $\dfrac{6\,ab^2}{9\,a^2b}.$
2. $\dfrac{3\,ab^2c}{15\,a^2b^2c^2}.$
3. $\dfrac{26\,x^2y^3}{39\,xy^5}.$
4. $\dfrac{42\,m^2b}{49\,mn^2}.$
5. $\dfrac{30\,xy^3z^4}{18\,x^2y^2z^2}.$
6. $\dfrac{21\,m^2n^2}{28\,m^2p}.$
7. $\dfrac{34\,ax^3y^2}{51\,a^2xy^7}.$
8. $\dfrac{35\,a^6b^4c^2}{5\,a^3b^3c}.$
9. $\dfrac{58\,ab^2c^3}{87\,a^4b^3c^2}.$

10. $\dfrac{9xy - 12y^2}{12x^2 - 16xy}$.

11. $\dfrac{4a^2 - 9c^2}{4a^2 + 6ac}$.

12. $\dfrac{3a^2 + 6a}{a^2 + 4a + 4}$.

13. $\dfrac{b^2 - 5b}{b^2 - 4b - 5}$.

14. $\dfrac{20(a^3 - c^3)}{4(a^2 + ac + c^2)}$.

15. $\dfrac{x^3 + y^3}{x^2 + 2xy + y^2}$.

16. $\dfrac{x^3 - 27}{x^2 + 2x - 15}$.

17. $\dfrac{x^2 - 8x + 15}{2x^2 - 13x + 21}$.

18. $\dfrac{x^2 - x - 20}{2x^2 - 7x - 15}$.

19. $\dfrac{4x^2 + 12ax + 9a^2}{8x^3 + 27a^3}$.

20. $\dfrac{x^2 - y^2 - 2yz - z^2}{x^2 + 2xy + y^2 - z^2}$.

21. $\dfrac{x^4 + x^2y^2 + y^4}{x^3 - y^3}$.

22. $\dfrac{2a^2 + 17a + 21}{3a^2 + 26a + 35}$.

23. $\dfrac{(a + b)^2 - c^2}{(a + b + c)^2}$.

24. $\dfrac{(x - a)^2 - b^2}{(x - b)^2 - a^2}$.

25. $\dfrac{(x + a)^2 - b^2}{(x + b)^2 - a^2}$.

26. $\dfrac{(a + b)^2 - (c + d)^2}{(a + c)^2 - (b + d)^2}$.

27. $\dfrac{(a + c)^2 - b^2}{4a^2c^2 - (a^2 + c^2 - b^2)^2}$.

Reduce by the Factor Theorem; or by finding the H.C.F. of the numerator and denominator:

28. $\dfrac{x^3 - 6x - 4}{3x^3 - 8x + 8}$.

29. $\dfrac{x^3 - 3x + 2}{x^3 + 4x^2 - 5}$.

30. $\dfrac{3x^3 - x^2 - x - 1}{3x^3 - 4x^2 - x + 2}$.

31. $\dfrac{x^4 - 13x^2 + 36}{x^4 - x^3 - 7x^2 + x + 6}$.

32. $\dfrac{3x^3 + 17x^2 + 22x + 8}{6x^3 + 25x^2 + 23x + 6}$.

33. $\dfrac{x^3 - 3x^2 - 15x + 25}{x^3 + 7x^2 + 5x - 25}$.

34. $\dfrac{2x^3 + x^2 - 8x + 3}{3x^3 + 8x^2 + x - 2}$.

35. $\dfrac{x^3 + 4x^2 - 8x + 24}{x^4 - x^3 + 8x - 8}$.

159. There are *three signs* to consider in a fraction, the sign before the fraction, the sign of the numerator, and the sign of the denominator.

Since, $$\frac{a}{b}=\frac{-a}{-b}=-\frac{-a}{b}=-\frac{a}{-b}, \qquad (§\ 87)$$

any *two* of the three signs may be changed without changing the value of the fraction.

The sign of a compound expression is changed by changing the *sign of every term* of the expression. Hence,

1. *We may change the sign of every term of the numerator and denominator of a fraction without changing the value of the fraction.*

2. *We may change the sign before a fraction and the sign of every term of either the numerator or denominator without changing the value of the fraction.*

160. From the Law of Signs, therefore,

1. We may change the signs of an **even number of factors** of the numerator, or of the denominator, or of both, without changing the sign of the fraction.

2. We may change the signs of an **odd number of factors** of the numerator, or of the denominator, or of both counted together, *if we change the sign before the fraction.*

Reduce to its lowest terms $\frac{(a-b)(c-d)}{(b-a)(c+d)}$.

Change the sign of the factor $(b-a)$ of the denominator and the sign before the fraction, and we have

$$\frac{(a-b)(c-d)}{(b-a)(c+d)}=-\frac{(a-b)(c-d)}{(a-b)(c+d)}=-\frac{c-d}{c+d}.$$

In the last fraction change the sign of the numerator, the sign of the fraction, and the *order* of the terms of the denominator, and we have

$$-\frac{c-d}{c+d}=\frac{d-c}{d+c}.$$

NOTE. Factors and terms must not be confounded.

EXERCISE 53.

Reduce to lowest terms:

1. $\dfrac{xy - 3y^2}{27y^3 - x^3}$.
2. $\dfrac{abx - bx^2}{cx^2 - acx}$.
3. $\dfrac{4a^2 - 6ab}{9b^2 - 4a^2}$.
4. $\dfrac{1 - x^2}{4x(x - 1)}$.
5. $\dfrac{x^2 + 5x - 14}{4 - x^2}$.
6. $\dfrac{3ab(b^2 - a^2)}{4(a^2b - ab^2)}$.
7. $\dfrac{c^2 - (a + b)^2}{a^2 + ab - ac}$.
8. $\dfrac{ac - bc - ax + bx}{x^2 - c^2}$.
9. $\dfrac{(b + c + d)^2 - a^2}{(a - b)^2 - (c + d)^2}$.
10. $\dfrac{(x - a)^2 - b^2}{(a - b)^2 - x^2}$.

161. Mixed Expressions. A mixed expression is an integral expression and a fraction.

Thus, $x + \dfrac{1}{x}$ and $x + y - \dfrac{x - y}{x + y}$ are mixed expressions.

Reduction of Fractions to Integral or Mixed Expressions.

By the distributive law of division, § 43,

$$\frac{a^2 + c}{a} = \frac{a^2}{a} + \frac{c}{a} = a + \frac{c}{a}.$$

Therefore, a fraction whose numerator is of a degree equal to, or higher than, the degree of the denominator is reduced to an integral or mixed expression by division. Hence,

162. To Reduce a Fraction to an Integral or Mixed Expression,

Divide the numerator by the denominator.

Thus, $\dfrac{10x^2 + 15x - 7}{5x} = \dfrac{10x^2}{5x} + \dfrac{15x}{5x} - \dfrac{7}{5x} = 2x + 3 - \dfrac{7}{5x}.$

Reduce $\dfrac{a^3 + 4a^2 - 5}{a^2 + a - 2}$ to an integral or mixed expression.

$$
\begin{array}{r|l}
a^3 + 4a^2 \qquad\quad - 5 & a^2 + a - 2 \\ \cline{2-2}
a^3 + a^2 - 2a \qquad & a + 3 \\ \cline{1-1}
3a^2 + 2a - 5 & \\
3a^2 + 3a - 6 & \\ \cline{1-1}
- \quad a + 1 &
\end{array}
$$

The remainder $-a + 1$ is the numerator of a fraction, and the divisor $a^2 + a - 2$ is its denominator, to be added to the integral quotient $a + 3$. Thus, the complete expression required is

$$a + 3 + \frac{-a + 1}{a^2 + a - 2}.$$

By changing the sign of each term of the numerator and the sign before the fraction,

$$a + 3 + \frac{-a + 1}{a^2 + a - 2} = a + 3 - \frac{a - 1}{a^2 + a - 2}.$$

The last form of the expression is the form usually written.

Exercise 54.

Reduce to an integral or a mixed expression:

1. $\dfrac{4x^2 + 12x - 3}{4x}$.

2 $\dfrac{3x^2 - 6x - 2}{3x}$.

3. $\dfrac{x^3 + y^3}{x - y}$.

4. $\dfrac{x^3 - y^3}{x + y}$.

5. $\dfrac{a^3 + 2x^3}{a + 2x}$.

6. $\dfrac{4x^2 - 3x - 54}{x - 4}$.

7. $\dfrac{3x^3 - x^2 - 16x - 2}{x^2 + x - 3}$.

8. $\dfrac{x^3 + 4x^2 - 7}{x^2 + x - 2}$.

9. $\dfrac{4x^2 + 6ax - 27a^2 - 2}{2x - 3a}$

10. $\dfrac{5x^3 + 9x^2 + 5}{5x^2 + 4x - 1}$.

Reduction of Mixed Expressions to Fractions.

163. The value of a number is unaltered if it is both multiplied and divided by the same number. Hence,

To Reduce a Mixed Expression to a Fraction,

Multiply the integral expression by the denominator, to the product add the numerator, and under the result write the denominator.

Reduce to a fraction $a - b - \frac{a^2 - ab - b^2}{a + b}$.

$$\begin{aligned} a - b - \frac{a^2 - ab - b^2}{a + b} &= \frac{(a - b)(a + b) - (a^2 - ab - b^2)}{a + b} \\ &= \frac{a^2 - b^2 - a^2 + ab + b^2}{a + b} \\ &= \frac{ab}{a + b}. \end{aligned}$$

NOTE. The dividing line between the terms of a fraction has the force of a vinculum affecting the numerator. If, therefore, a *minus sign* precedes the dividing line, as in the preceding example, and this line is removed, the numerator of the given fraction must be enclosed in a parenthesis preceded by the minus sign, or the sign of every term of the numerator must be changed.

EXERCISE 55.

Reduce to a fraction :

1. $a + b - \frac{2ab}{a + b}$.
2. $a + x - \frac{a^2 + x^2}{a + x}$.
3. $x + 4 - \frac{x - 12}{x - 3}$.
4. $a^2 - ax + x^2 - \frac{x^3}{a + x}$.
5. $a^2 + ax + x^2 - \frac{a^3}{a - x}$.
6. $\frac{a - 3x}{4} - a + 2x$.
7. $3a - 2b - \frac{3a^2 - 2b^2}{a + b}$.
8. $2x - 7 - \frac{21 - 13x}{x - 3}$.

Reduction of Fractions to Equivalent Fractions Having the Lowest Common Denominator.

1. Reduce $\frac{3x}{4a^2}$; $\frac{2y}{3a}$; $\frac{5}{6a^3}$ to equivalent fractions having the lowest common denominator.

The L. C. M. of the denominators is $12a^3$. (§ 152)

Divide the L. C. M. by the denominators $4a^2$; $3a$; $6a^3$.

The respective quotients are $3a$; $4a^2$; and 2.

Multiply both terms of the given fractions taken in order by the respective quotients, $3a$; $4a^2$; and 2.

We have for the required fractions

$$\frac{9ax}{12a^3};\ \frac{8a^2y}{12a^3};\ \text{and}\ \frac{10}{12a^3}.$$

2. Reduce $\frac{2}{x^2+5x+6}$; $\frac{3}{x^2+4x+3}$ to equivalent fractions having the lowest common denominator.

Express the denominators in their prime factors.

$$\frac{2}{x^2+5x+6};\ \frac{3}{x^2+4x+3} = \frac{2}{(x+3)(x+2)};\ \frac{3}{(x+3)(x+1)}.$$

The lowest common denominator (L. C. D.) is

$$(x+3)(x+2)(x+1).$$

The respective quotients are $x+1$; $x+2$.

The respective products are $2(x+1)$; $3(x+2)$.

The required fractions are

$$\frac{2(x+1)}{(x+3)(x+2)(x+1)};\ \frac{3(x+2)}{(x+3)(x+2)(x+1)}.$$

164. Therefore, we have the following rule:

Find the lowest common multiple of the denominators of the given fractions for the common denominator. Divide this common denominator by each of the given denominators; and multiply the given numerators each by the corresponding quotient for the required numerators.

Note. Every fraction should be in its lowest terms before the common denominator is found.

EXERCISE 56.

Express with lowest common denominator:

1. $\frac{3x-7}{6};\ \frac{4x-9}{18}.$

2. $\frac{a-2x}{3a};\ \frac{3x^2-2a}{9ax}.$

3. $\frac{2x-4y}{5x^2};\ \frac{3x-8y}{10x}.$

4. $\frac{4a-5c}{5ac};\ \frac{3a-2c}{12a^2c}.$

5. $\frac{5}{1-x};\ \frac{6}{1-x^2}.$

6. $\frac{1}{x+2};\ \frac{2}{x+3}.$

7. $\frac{a}{x-a};\ \frac{a^2}{x^2-a^2}.$

8. $\frac{1}{1+2a};\ \frac{1}{1-4a^2}.$

9. $\frac{9}{16-x^2};\ \frac{4-x}{4+x}.$

10. $\frac{a^2}{27-a^3};\ \frac{a}{3-a}.$

11. $\frac{4a^2+c^2}{4a^2-c^2};\ \frac{2a+c}{2a-c}.$

12. $\frac{x^2+y^2}{25x^2-4y^2};\ \frac{1}{5x+2y}.$

13. $\frac{x+2}{x-2};\ \frac{x-2}{x+2}.$

14. $\frac{4x^2}{3(a+b)};\ \frac{xy}{6(a^2-b^2)}.$

15. $\frac{8x+2}{x-2};\ \frac{2x-1}{3x-6};\ \frac{3x+2}{5x-10}.$

16. $\frac{a-bm}{4x};\ 1;\ \frac{c-bn}{3x}.$

17. $\frac{1}{(a-b)(b-c)};\ \frac{1}{(a-b)(a-c)}.$

18. $\frac{x}{x-1};\ \frac{x^2}{x+1};\ \frac{x^3}{x^2-1}.$

19. $\frac{a}{a-b};\ \frac{b}{a+b};\ \frac{c}{a^2-b^2}.$

20. $\frac{1}{x+1};\ \frac{x}{(x+1)^2};\ \frac{x^2}{(x+1)^2}.$

21. $\frac{1}{(x-y)(x-z)};\ \frac{1}{(x-y)(x-c)};\ \frac{1}{(x-c)(x-z)}.$

22. $\frac{1}{x^2-5x+6};\ \frac{1}{x^2-4x+3};\ \frac{1}{x^2-3x+2}.$

23. $\frac{1}{x-2a};\ \frac{1}{x^2-5ax+6a^2};\ \frac{1}{x-3a}$

Addition and Subtraction of Fractions.

1. Find the algebraic sum of $\frac{a}{x}+\frac{b}{x}-\frac{c}{x}$.

$$\frac{a}{x}+\frac{b}{x}-\frac{c}{x}=\frac{a+b-c}{x}. \qquad (\S\ 43)$$

2. Find the algebraic sum of

$$\frac{3a-4b}{4}-\frac{2a-b+c}{3}+\frac{a-4c}{12}.$$

The L. C. D. = 12.

The multipliers, that is, the quotients obtained by dividing 12 by 4, 3, and 12, are 3, 4, and 1, respectively.

Hence, the sum of the fractions is

$$\begin{aligned}&\frac{9a-12b}{12}-\frac{8a-4b+4c}{12}+\frac{a-4c}{12}\\ &=\frac{9a-12b-8a+4b-4c+a-4c}{12}\\ &=\frac{2a-8b-8c}{12}=\frac{a-4b-4c}{6}.\end{aligned}$$

The preceding work may be arranged as follows:

The L. C. D. = 12.

The multipliers are 3, 4, and 1, respectively.

$$\begin{array}{rcll}3(3a-4b) &=& 9a-12b & = \text{1st numerator.}\\ -4(2a-b+c) &=& -8a+4b-4c & = \text{2d numerator.}\\ 1(a-4c) &=& \underline{a \qquad\quad -4c} & = \text{3d numerator.}\\ && 2a-8b-8c & \end{array}$$

or $2(a-4b-4c)=$ the sum of the numerators.

$$\therefore \text{sum of fractions}=\frac{2(a-4b-4c)}{12}=\frac{a-4b-4c}{6}.$$

165. To Add Fractions, therefore,

Reduce the fractions, if they have different denominators, to equivalent fractions having the lowest common denominator; and write the algebraic sum of the numerators of these fractions over the common denominator.

NOTE. The resulting fraction should be expressed in its lowest terms.

EXERCISE 57.

Find the algebraic sum of:

1. $\frac{x-1}{2}-\frac{x-3}{5}-\frac{x-7}{10}+\frac{x-2}{5}$.

2. $\frac{2x-1}{3}-\frac{x+7}{6}+\frac{x-4}{4}-\frac{x-3}{2}$.

3. $\frac{7x-5}{8}-\frac{3x+2}{3}+\frac{x+1}{4}-\frac{5x-10}{12}$.

4. $\frac{2x+3}{9}+\frac{x-2}{6}-\frac{5x+4}{12}-\frac{2x-4}{3}$.

5. $\frac{2x+3}{2x}+\frac{x+3}{4x}-\frac{18x+5}{8x^2}-\frac{x-3}{x}$.

6. $\frac{x}{2}-\frac{2x-11}{3}-\frac{x+3}{4}+\frac{x-7}{6}-\frac{x-1}{12}$.

7. $\frac{4a^2}{b^2}-\frac{a+b}{ab}+\frac{4b^2}{a^2}+\frac{a^2b+ab^2-4a^4}{a^2b^2}$.

8. $\frac{5x-11}{4}-\frac{x-1}{10}+\frac{11x-1}{12}-\frac{12x-5}{3}$.

9. $\frac{x+1}{2}+\frac{3x-4}{5}+\frac{1}{4}-\frac{6x+7}{8}$.

10. $\frac{2x-6}{5x}-\frac{8x-4}{15x}+\frac{56x-48}{45x}$.

11. $\frac{11xy+2}{x^2y^2}-\frac{5y^2-3}{xy^3}-\frac{6x^2-5}{x^3y}$.

12. $\frac{a-b}{c}+\frac{b-c}{a}+\frac{c-a}{b}+\frac{ab^2+bc^2+ca^2}{abc}$.

13. $\frac{3}{2x^2y}+\frac{1}{6y^2z}-\frac{1}{2xz^2}+\frac{2x-z}{4x^2z^2}+\frac{y-6z}{4x^2yz}$

166. When the denominators are polynomials arranged in the same order, we first express the denominators in their prime factors.

Find the algebraic sum of:

$$\frac{a+b}{a-b}-\frac{a-b}{a+b}-\frac{4\,ab}{a^2-b^2}.$$

$$a^2-b^2=(a+b)(a-b).$$

The L.C.D. is $(a-b)(a+b)$.

The multipliers are $a+b$, $a-b$, and 1, respectively.

$$\begin{array}{lcll}
(a+b)(a+b) & = & a^2+2\,ab+b^2 & = \text{1st numerator.} \\
-(a-b)(a-b) & = & -a^2+2\,ab-b^2 & = \text{2d numerator.} \\
-1(4\,ab) & = & \underline{\quad -4\,ab \quad} & = \text{3d numerator.} \\
 & & 0 & = \text{sum of numerators.}
\end{array}$$

$\therefore$ sum of fractions $=0$.

Exercise 58.

Find the algebraic sum of:

1. $\dfrac{1}{x+6}+\dfrac{1}{x-5}.$

2. $\dfrac{1}{1+x}-\dfrac{1}{1-x}.$

3. $\dfrac{1}{1+x}-\dfrac{2}{1-x^2}.$

4. $\dfrac{x+y}{x-y}-\dfrac{x^2-y^2}{(x-y)^2}.$

5. $\dfrac{x-y}{x+y}-\dfrac{(x-y)^2}{(x+y)^2}.$

6. $\dfrac{1}{2\,a^2+2\,ab}+\dfrac{1}{2\,a^2-2\,ab}.$

7. $\dfrac{1}{a-3\,c}-\dfrac{(a-3\,c)^2}{a^3-27\,c^3}.$

8. $\dfrac{1+x}{1+x+x^2}-\dfrac{1-x}{1-x+x^2}.$

9. $\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}-\dfrac{4\,xy}{x^2-y^2}.$

10. $\dfrac{x}{a-x}+\dfrac{x}{a+x}+\dfrac{2\,x^2}{a^2+x^2}.$

11. $\dfrac{1}{x}-\dfrac{2}{x+1}+\dfrac{1}{x+2}.$

12. $\dfrac{6-2\,a}{9-a^2}-\dfrac{2}{3+a}-\dfrac{1}{3-a}.$

13. $\dfrac{b}{a+b}-\dfrac{ab}{(a+b)^2}-\dfrac{ab^2}{(a+b)^3}.$

14. $\dfrac{x+y}{y}-\dfrac{2x}{x+y}-\dfrac{x^2(x-y)}{y(x^2-y^2)}.$

Hint. Reduce the last fraction to lowest terms.

15. $\dfrac{1}{x+2y}+\dfrac{1}{x-2y}-\dfrac{x}{x^2-4y^2}.$

16. $\dfrac{1}{y-1}-\dfrac{2}{y}+\dfrac{1}{y+1}-\dfrac{1}{y^2-1}.$

17. $\dfrac{x}{x-1}-1-\dfrac{1}{x^2-x}+\dfrac{1}{x}.$

18. $\dfrac{3}{x-a}+\dfrac{4a}{(x-a)^2}-\dfrac{5a^2}{(x-a)^3}.$

19. $\dfrac{1}{x-2a}+\dfrac{a^2}{x^3-8a^3}-\dfrac{x+a}{x^2+2ax+4a^2}.$

20. $\dfrac{x^2-2x+3}{x^3+1}+\dfrac{x-2}{x^2-x+1}-\dfrac{1}{x+1}.$

21. $\dfrac{1}{x-3}+\dfrac{x-1}{x^2+3x+9}+\dfrac{x^2+x-3}{x^3-27}.$

22. $\dfrac{x^2+8x+15}{x^2+7x+10}-\dfrac{x-1}{x-2}.$

HINT. Reduce the first fraction to lowest terms.

23. $\dfrac{x^2-5ax+6a^2}{x^2-8ax+15a^2}-\dfrac{x-7a}{x-5a}.$

24. $\dfrac{1}{x-2}+\dfrac{1}{x^2-3x+2}-\dfrac{2}{x^2-4x+3}.$

HINT. Express the denominators of the last two fractions in prime factors.

25. $\dfrac{1}{a^2-7a+12}+\dfrac{2}{a^2-4a+3}-\dfrac{3}{a^2-5a+4}.$

26. $\dfrac{3}{10a^2+a-3}-\dfrac{4}{2a^2+7a-4}.$

27. $\dfrac{3}{2-x-6x^2}-\dfrac{1}{1-x-2x^2}.$

167. When the denominators are polynomials not arranged in the same order, we first write the fractions, by § 159, so that the denominators shall be arranged in the same order.

1. Find the sum of $\frac{2}{x} - \frac{3}{2x-1} + \frac{2x-3}{1-4x^2}$.

Change the signs before the terms of the denominator of the third fraction and the sign before the fraction. We now have

$$\frac{2}{x} - \frac{3}{2x-1} - \frac{2x-3}{4x^2-1}.$$

Proceeding as in § 166, we find for the required sum

$$\frac{-2}{x(2x-1)(2x+1)} = \frac{2}{x(1-2x)(1+2x)},$$

if we change the sign of 2 and the signs of $(2x-1)$, § 160.

2. Find the sum of

$$\frac{1}{a(a-b)(a-c)} + \frac{1}{b(b-a)(b-c)} + \frac{1}{c(c-a)(c-b)}.$$

Change the sign of the factor $(b-a)$ in the denominator of the second fraction, and change the sign before the fraction, § 160.

Change the signs of the two factors $(c-a)$ and $(c-b)$ in the denominator of the third fraction, § 160. We now have

$$\frac{1}{a(a-b)(a-c)} - \frac{1}{b(a-b)(b-c)} + \frac{1}{c(a-c)(b-c)}.$$

The L.C.D. $= abc(a-b)(a-c)(b-c)$.

$$\begin{array}{rl} bc(b-c) = b^2c - bc^2 & = \text{1st numerator.} \\ -ac(a-c) = -a^2c + ac^2 & = \text{2d numerator.} \\ \underline{ab(a-b) = a^2b - ab^2} & = \text{3d numerator.} \\ a^2b - a^2c - ab^2 + ac^2 + b^2c - bc^2 & = \text{sum of numerators.} \\ = a^2(b-c) - a(b^2-c^2) + bc(b-c). & \end{array}$$

Divide by the common factor $(b-c)$, and we have

$$a^2 - ab - ac + bc;$$

and this is equal to $(a-b)(a-c)$.

$\therefore (a-b)(a-c)(b-c) =$ sum of numerators.

$\therefore$ sum of fractions $= \frac{(a-b)(a-c)(b-c)}{abc(a-b)(a-c)(b-c)} = \frac{1}{abc}.$

EXERCISE 59.

Find the algebraic sum of:

1. $\frac{x^2}{x^2-1}+\frac{x}{x+1}-\frac{x}{1-x}.$

2. $\frac{a}{a-x}+\frac{3a}{a+x}+\frac{2ax}{x^2-a^2}.$

3. $\frac{3}{2a-3}-\frac{2}{3+2a}+\frac{15}{9-4a^2}.$

4. $\frac{a-b}{b}+\frac{2a}{a-b}+\frac{a^3+a^2b}{b^3-a^2b}.$

5. $\frac{3}{x}+\frac{5}{1-2x}-\frac{2x-7}{4x^2-1}.$

6. $\frac{1}{(x+a)^2}+\frac{1}{(a-x)^2}+\frac{1}{x^2-a^2}.$

7. $\frac{1}{x-y}+\frac{x-y}{x^2+xy+y^2}-\frac{xy-2x^2}{y^3-x^3}.$

8. $\frac{1}{(x-2)(x-3)}+\frac{2}{(x-1)(3-x)}+\frac{1}{(x-1)(x-2)}$

9. $\frac{bc}{(c-a)(a-b)}+\frac{ac}{(a-b)(b-c)}+\frac{ab}{(b-c)(c-a)}.$

10. $\frac{b+c}{(a-b)(a-c)}+\frac{a+c}{(b-c)(b-a)}+\frac{a+b}{(c-a)(c-b)}.$

11. $\frac{3}{(a-b)(b-c)}-\frac{4}{(b-a)(c-a)}-\frac{6}{(a-c)(c-b)}.$

12. $\frac{1}{x(x-y)(x-z)}+\frac{1}{y(y-x)(y-z)}-\frac{1}{xyz}.$

13. $\frac{a^2-bc}{(a-b)(a-c)}+\frac{b^2+ac}{(b-a)(b+c)}+\frac{c^2+ab}{(c-a)(c+b)}.$

Multiplication of Fractions.

Find the product of $\frac{a}{b} \times \frac{c}{d}$.

Let $$\frac{a}{b} \times \frac{c}{d} = x. \tag{1}$$

Multiply each of these equals by $b \times d$.
Since the order of the factors is immaterial, (§ 42)

$$\left(\frac{a}{b} \times b\right) \times \left(\frac{c}{d} \times d\right) = b \times d \times x.$$

Or, $$a \times c = b \times d \times x.$$

Divide by $b \times d$, $$\frac{a \times c}{b \times d} = x. \tag{2}$$

From (1) and (2), $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$.

Likewise, $$\frac{a}{b} \times \frac{c}{d} \times \frac{e}{f} = \frac{a \times c}{b \times d} \times \frac{e}{f} = \frac{a \times c \times e}{b \times d \times f};$$

and so on for any number of fractions. Hence,

168. To Find the Product of Two or More Fractions,

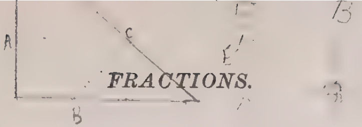
Find the product of the numerators for the required numerator, and of the denominators for the required denominator.

In applying the rule, reduce every mixed expression to a fraction, and every integral expression to a fraction with 1 for the denominator.

Cancel every factor common to a numerator and a denominator, as the cancelling of a common factor *before* the multiplication is equivalent to cancelling it *after* the multiplication.

1. Find the product of $\frac{2a^2b}{3cd^2} \times \frac{6c^2d}{5ab} \times \frac{5ab^2c}{8a^2c^2d^2}$.

$$\frac{2a^2b}{3cd^2} \times \frac{6c^2d}{5ab} \times \frac{5ab^2c}{8a^2c^2d^2} = \frac{2 \times 6 \times 5a^3b^3c^3d}{3 \times 5 \times 8a^3bc^3d^4} = \frac{b^2}{2d^3}.$$



2. Find the product of

$$\frac{x^2-y^2}{x^2-3xy+2y^2}\times\frac{xy-2y^2}{x^2+xy}\times\frac{x^2-xy}{(x-y)^2}.$$

Express the numerators and denominators in prime factors.

$$\frac{(x-y)(x+y)}{(x-y)(x-2y)}\times\frac{y(x-2y)}{x(x+y)}\times\frac{x(x-y)}{(x-y)(x-y)}=\frac{y}{x-y}.$$

The common factors cancelled are $(x-y)$, $(x+y)$, $(x-2y)$, x, and $(x-y)$.

Exercise 60.

Find the product of:

1. $\frac{12x^2}{7y^2}\times\frac{14xy}{9x^2z^3}$.
2. $\frac{3a^2b^2c^3}{4amn}\times\frac{20m^2n^3}{21a^4c^5}$.
3. $\frac{6a^2b^2c^5}{7mxy}\times\frac{5m^2x^3}{3a^3c^6}$.
4. $\frac{9m^2n^2}{8x^3y^3}\times\frac{4x^2y^2}{15mn}$.
5. $\frac{16a^4b^2}{21x^3y^2}\times\frac{3x^2y}{4a^2b}$.
6. $\frac{7xy}{12z^2}\times\frac{36yz^2}{35xz}$.
7. $\frac{x^2-y^2}{x^2+y^2}\times\frac{4x}{x+y}$.
8. $\frac{3x^2-x}{a}\times\frac{2a}{2x^2-4x}$.
9. $\frac{3x^2}{5x-10}\times\frac{3x-6}{4x^3}$.
10. $\frac{8a^3}{a^3-b^3}\times\frac{a^2+ab+b^2}{4a^2}$.
11. $\frac{x^2+y^2}{x^2-y^2}\times\frac{x+y}{3x^2+3y^2}$.
12. $\frac{ab-b^2}{a(a+b)}\times\frac{a(a^2-b^2)}{b^2}$.
13. $\frac{a^2-4x^2}{a^2+4ax}\times\frac{ax+4x^2}{a^2-2ax}$.
14. $\frac{x^4-y^4}{(x-y)^2}\times\frac{x-y}{x^2+xy}$.
15. $\frac{x^3+a^3}{x^2-9a^2}\times\frac{x+3a}{x+a}$.
16. $\frac{a^3-b^3}{a^3+b^3}\times\frac{a^2-ab+b^2}{a-b}$.
17. $\frac{x^2-1}{x^2-4x-5}\times\frac{x^2-25}{x^2+2x-3}$.
18. $\left(1-\frac{y^4}{x^4}\right)\times\left(\frac{xy}{x^2+y^2}\right)$.

19. $\left(\frac{4x^2}{y^2}-1\right)\left(\frac{2x}{2x-y}-1\right).$

20. $\left(\frac{8x^3}{y^3}-1\right)\left(\frac{4x^2+2xy}{4x^2+2xy+y^2}-1\right).$

21. $\left(x-\frac{xy-y^2}{x+y}\right)\left(x-\frac{xy^2-y^3}{x^2+y^2}\right)\times\frac{x^2}{x^2-xy+y^2}.$

22. $\frac{8a^2b}{c}\times\frac{c^2d}{8a^3}\times\frac{4ab}{cd}\times\frac{bcd-cd^2}{4(b^2-bd)}.$

23. $\frac{y(x^3-y^3)}{x(x+y)}\times\frac{(x^2-y^2)^2}{x^2+xy+y^2}\times\frac{(x+y)^2}{(x-y)^2}.$

24. $\frac{(a+b)^2-c^2}{a^2+ab-ac}\times\frac{a}{(a+c)^2-b^2}\times\frac{(a-b)^2-c^2}{ab-b^2-bc}.$

25. $\frac{(a+b)^2-c^2}{a^2-(b-c)^2}\times\frac{c^2-(a-b)^2}{c^2-(a+b)^2}\times\frac{c-a-b}{ac-a^2+ab}.$

26. $\frac{(x-a)^2-b^2}{(x-b)^2-a^2}\times\frac{x^2-(b-a)^2}{x^2-(a-b)^2}\times\frac{ax+a^2-ab}{bx-ab+b^2}$

27. $\frac{a^2-2ab+b^2-c^2}{a^2+2ab+b^2-c^2}\times\frac{a+b-c}{a-b+c}.$

28. $\frac{x^2+(x+1)^2}{x(x+1)}\times\frac{x^2+x}{(x+1)^2-x^2}\times\frac{2x+1}{x}.$

29. $\frac{2ax^3+2a^3x}{(x-a)^2(x+a)^2}\times\frac{x^2-a^2}{2(x^2+a^2)}\times\frac{x+a}{ax}.$

30. $\frac{a^2-b^2-c^2-2bc}{a^2-ab-ac}\times\left(1-\frac{2c}{a+b+c}\right).$

31. $\frac{a^4+a^2b^2+b^4}{a^6-b^6}\times\frac{a+b}{a^3+b^3}\times\frac{a^2-b^2}{a}.$

32. $\frac{x^2+7xy+12y^2}{x^2+5xy+6y^2}\times\frac{x^2+xy-2y^2}{x^2+3xy-4y^2}.$

Division of Fractions.

169. Reciprocals. If the product of two numbers is equal to 1, each of the numbers is called the **reciprocal** of the other.

The reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$; for $\frac{a}{b} \times \frac{b}{a} = \frac{ab}{ab} = 1.$

The reciprocal of a fraction, therefore, is the fraction inverted.

Find the quotient of $\frac{a}{b} \div \frac{c}{d}$.

Let $$\frac{a}{b} \div \frac{c}{d} = x. \qquad (1)$$

Since the dividend is the product of the divisor and quotient,

$$\frac{a}{b} = \frac{c}{d} \times x.$$

Multiply each of these equals by $\frac{d}{c}$.

Then $$\frac{a}{b} \times \frac{d}{c} = \frac{c}{d} \times \frac{d}{c} \times x = x. \qquad (2)$$

From (1) and (2), $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$. Therefore,

170. To Divide by a Fraction,

Multiply the dividend by the reciprocal of the divisor.

Find the result of $\frac{1}{x} \times \frac{x^2-1}{x^2-4x-5} \div \frac{x^2+2x-3}{x^2-25}$.

$$\frac{1}{x} \times \frac{x^2-1}{x^2-4x-5} \div \frac{x^2+2x-3}{x^2-25} = \frac{1}{x} \times \frac{x^2-1}{x^2-4x-5} \times \frac{x^2-25}{x^2+2x-3}$$

$$= \frac{1}{x} \times \frac{(x-1)(x+1)}{(x-5)(x+1)} \times \frac{(x-5)(x+5)}{(x+3)(x-1)} = \frac{x+5}{x(x+3)}$$

The common factors cancelled are $(x-1)$, $(x+1)$, $(x-5)$.

EXERCISE 61.

Find the quotient of:

1. $\dfrac{15x^2}{7y^2} \div \dfrac{9x^3z}{28xy}.$

2. $\dfrac{3x^2y^2z^2}{4a^2b^2c^2} \div \dfrac{18x^2yz^2}{9a^3b^2c}.$

3. $\dfrac{5a^2b^2c^4}{3m^2np^4} \div \dfrac{20m^2n^2p^3}{21abc^4}.$

4. $\dfrac{16a^3b^2c^4}{21m^3x^3y^4} \div \dfrac{8a^2b^2c^3}{7m^3x^3y^4}.$

5. $\dfrac{2a^2}{bc} \div \dfrac{ac}{3b^2} \times \dfrac{ab}{5c^2}.$

6. $\dfrac{5c^3}{2ab} \times \dfrac{3b^3}{5ac} \div \dfrac{3bc}{2a^3}.$

7. $\dfrac{x^2-a^2}{x^2-4a^2} \div \dfrac{x-a}{x+2a}.$

8. $\dfrac{x^2y^2+3xy}{4c^2-1} \div \dfrac{xy+3}{2c-1}.$

9. $\dfrac{9x^2-4y^2}{4-x^2} \div \dfrac{3x-2y}{2+x}.$

10. $\dfrac{x^2-4}{x^2+4x} \div \dfrac{x^2+2x}{x^2-16}.$

11. $\dfrac{a^2-4a+3}{a^2-5a+4} \div \dfrac{a^2-10a+21}{a^2-9a+20} \times \dfrac{a^2-7a}{a^2-5a}.$

12. $\dfrac{b^2-7b+6}{b^2+3b-4} \div \dfrac{b^2-14b+48}{b^2+10b+24} \div \dfrac{b^2+6b}{b^3-8b^2}.$

13. $\dfrac{x^2-y^2}{x^2-3xy+2y^2} \div \dfrac{x^2+xy}{xy-2y^2} \times \dfrac{x^2-xy}{(x-y)^2}.$

14. $\dfrac{(a-b)^2}{a^2-b^2} \div \dfrac{2ab-2b^2}{3} \times \dfrac{a^2+ab}{a-b}.$

15. $\dfrac{(a+b)^2-c^2}{a^2-(b-c)^2} \div \dfrac{c^2-(a+b)^2}{c^2-(a-b)^2}.$

16. $\dfrac{(x-a)^2-b^2}{(x-b)^2-a^2} \div \dfrac{x^2-(a-b)^2}{x^2-(b-a)^2}.$

17. $\dfrac{(a+b)^2-(c+d)^2}{(a+c)^2-(b+d)^2} \div \dfrac{(a-c)^2-(d-b)^2}{(a-b)^2-(d-c)^2}.$

18. $\dfrac{x^2-2xy+y^2-z^2}{x^2+2xy+y^2-z^2} \div \dfrac{x-y+z}{x+y-z}.$

Complex Fractions.

171. A complex fraction is a fraction that has one or more fractions in either or both of its terms.

Simplify the complex fraction $\dfrac{3x}{\dfrac{4x-1}{4}}$.

$$\frac{3x}{\dfrac{4x-1}{4}} = 3x \div \frac{4x-1}{4} = 3x \times \frac{4}{4x-1} = \frac{12x}{4x-1}.$$ Hence,

172. To Simplify a Complex Fraction,

Divide the numerator by the denominator.

173. The shortest way to simplify a complex fraction is to multiply both terms of the fraction by the L. C. D. of the fractions contained in the numerator and denominator, § 157.

1. Simplify $\dfrac{\dfrac{a}{a-x} - \dfrac{a}{a+x}}{\dfrac{x}{a-x} + \dfrac{x}{a+x}}$.

The L. C. M. of $a - x$ and $a + x$ is $(a - x)(a + x)$.
Multiply both terms by $(a - x)(a + x)$, and we have

$$\frac{a(a+x) - a(a-x)}{x(a+x) + x(a-x)} = \frac{a^2 + ax - a^2 + ax}{ax + x^2 + ax - x^2} = \frac{2ax}{2ax} = 1.$$

2. Simplify $\dfrac{1}{1 + \dfrac{1}{1 + \dfrac{1}{1-x}}}$.

Multiply both terms of the *last* complex fraction by $1 - x$. We have $\dfrac{1-x}{2-x}$, and this put in place of the last complex fraction changes the given fraction to the form $\dfrac{1}{1 + \dfrac{1-x}{2-x}}$.

Multiply both terms of this fraction by $2 - x$. We have $\dfrac{2-x}{3-2x}$.

EXERCISE 62.

Simplify:

1. $$\frac{\frac{x}{m}+\frac{y}{m}}{\frac{z}{m}}.$$

2. $$\frac{x+\frac{y}{z}}{z-\frac{y}{x}}.$$

3. $$\frac{\frac{ab}{c}-3d}{3c-\frac{ab}{d}}.$$

4. $$\frac{1+\frac{1}{x-1}}{1-\frac{1}{x+1}}.$$

5. $$\frac{1+\frac{y}{x-y}}{1-\frac{y}{x+y}}.$$

6. $$\frac{m+\frac{mn}{m-n}}{m-\frac{mn}{m+n}}.$$

7. $$\frac{1}{c-\frac{1}{c+\frac{1}{c}}}.$$

8. $$\frac{\frac{2m+n}{m+n}-1}{1-\frac{n}{m+n}}.$$

9. $$\frac{\frac{x^3+y^3}{x^2-y^2}}{\frac{x^2-xy+y^2}{x-y}}.$$

10. $$\frac{\frac{1}{a-b}-\frac{a}{a^2-b^2}}{\frac{a}{ab+b^2}-\frac{b}{a^2+ab}}.$$

✗ 11. $$\frac{\frac{1}{ab}+\frac{1}{ac}+\frac{1}{bc}}{\frac{a^2-(b+c)^2}{ab}}.$$

12. $$\frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}{\frac{a}{b}+\frac{b}{c}+\frac{c}{a}}.$$

✓ 13. $$\frac{\frac{1}{ab}-\frac{1}{ac}-\frac{1}{bc}}{\frac{a^2-(b-c)^2}{a}}.$$

14. $$\frac{1}{x-\frac{x^2-1}{x+\frac{1}{x-1}}}.$$

15. $$\frac{1}{a+\dfrac{1}{1+\dfrac{a+1}{3-a}}}.$$

16. $$\frac{x+y}{x+y+\dfrac{1}{x-y+\dfrac{1}{x+y}}}.$$

divide

divide

add

17. $$\frac{x+\dfrac{1}{y}}{x+\dfrac{1}{y+\dfrac{1}{z}}}-\frac{1}{y\,(xyz+x+z)}.$$

18. $$\frac{3\,abc}{bc+ac+ab}-\frac{\dfrac{a-1}{a}+\dfrac{b-1}{b}+\dfrac{c-1}{c}}{\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}}.$$

19. $$\left(\frac{m+n}{m-n}+\frac{m^2+n^2}{m^2-n^2}\right)\div\left(\frac{m-n}{m+n}-\frac{m^2+n^2}{m^2-n^2}\right).$$

20. $$\left(\frac{x-y}{x+y}-\frac{x^3-y^3}{x^3+y^3}\right)\times\left(\frac{x+y}{x-y}+\frac{x^2+y^2}{x^2-y^2}\right).$$

21. $$\frac{\dfrac{1}{a}+\dfrac{1}{b+c}}{\dfrac{1}{a}-\dfrac{1}{b+c}}\left(1+\frac{b^2+c^2-a^2}{2\,bc}\right).$$

22. $$\frac{x}{1-\dfrac{x}{1+x+\dfrac{x}{1-x+x^2}}}.$$

23. $$\frac{\dfrac{x^2-y^2-z^2-2\,yz}{x^2-y^2-z^2+2\,yz}}{\dfrac{x-y-z}{x+y-z}}.$$

EXERCISE 63. — REVIEW.

When $a = 2$, $b = -2$, and $c = 4$; find the value of:

1. $\sqrt{a^3 + b^3 + c^3} - (a - b - c)^2.$

2. $\dfrac{a^2 + b^2 - c^2 + 2\,ab}{a^2 - b^2 - c^2 + 2\,bc}.$

3. $(b - 5)(b - 4) - 3(b - 2)(b - 1) + 3(b + 1)(b + 2).$

4. Reduce to lowest terms $\dfrac{3\,x^3 + 10\,x^2 + 7\,x - 2}{3\,x^3 + 13\,x^2 + 17\,x + 6}.$

Simplify:

5. $\dfrac{1}{(x-3)(x-2)} - \dfrac{x-4}{(x-1)(x-3)} + \dfrac{x-3}{(x-1)(x-2)}.$

6. $\dfrac{a}{a-b} + \dfrac{a}{a+b} + \dfrac{2\,a^2}{a^2+b^2} + \dfrac{4\,a^4}{a^4+b^4}.$

HINT. Add the first two fractions; then their sum and the third fraction and this result to the fourth fraction.

7. $\dfrac{1}{a^2-2} - \dfrac{1}{a^2+2} + \dfrac{2}{a^2+1} - \dfrac{2}{a^2-1}.$

8. $\dfrac{1}{x+y} \div \left\{\left(\dfrac{1}{x+y} + \dfrac{1}{x-y}\right) \div \dfrac{2}{x+y}\right\}.$

9. $\dfrac{x^2 - 2\,x + 1}{(x-2)^2} \div \left\{\dfrac{x^2-1}{x^2-4} \times \dfrac{x+2}{x+1}\right\}.$

10. $\dfrac{a^2 - b^2}{a^2 - 3\,ab + 2\,b^2} \times \dfrac{ab - 2\,b^2}{a^2 + ab} \div \dfrac{(a-b)^2}{a(a-b)}.$

11. $\dfrac{(a+b)^2 - c^2}{a^2 + ab - ac} \times \dfrac{a^2b^2c^2}{a^2 + ab + ac} \div \dfrac{b^2c^2}{abc}.$

12. $\dfrac{x^2 + 7\,xy + 10\,y^2}{x^2 + 6\,xy + 5\,y^2} \times \dfrac{x+1}{x^2 + 4\,x + 4} \div \dfrac{1}{x+2}.$

13. $\dfrac{x^2+yz}{(x-y)(x-z)}+\dfrac{y^2+xz}{(y-z)(y-x)}+\dfrac{z^2+xy}{(z-x)(z-y)}.$

14. $\left(\dfrac{x}{1+x}+\dfrac{1-x}{x}\right)\div\left(\dfrac{x}{1+x}-\dfrac{1-x}{x}\right).$

15. $\dfrac{1}{c}\left(\dfrac{1}{x-c}+\dfrac{1}{x+2c}\right)-\dfrac{3}{x^2+cx-2c^2}.$

16. $\dfrac{x^4-y^4}{x^2y^2}\left(\dfrac{x^2}{x^2-y^2}-1+\dfrac{y^2}{x^2+y^2}\right).$

17. $\dfrac{3(x^2+x-2)}{x^2-x-2}-\dfrac{3(x^2-x-2)}{x^2+x-2}-\dfrac{8x}{x^2-4}.$

(x − 2)(x + 1) (x + 2)(x + 2

18. $\left(\dfrac{x+2y}{x+y}+\dfrac{x}{y}\right)\div\left(\dfrac{x}{y}+2-\dfrac{x}{x+y}\right).$

19. $\left(1-\dfrac{4}{x-1}+\dfrac{12}{x-3}\right)\left(1+\dfrac{4}{x+1}-\dfrac{12}{x+3}\right).$

20. $\dfrac{x^2-xy+y^2}{x^2+xy+y^2}\times\dfrac{x^3-y^3}{x^3+y^3}\div\dfrac{(y-x)^2}{(x+y)^2}.$

21. $\dfrac{2a-b-c}{(a-b)(a-c)}+\dfrac{2b-c-a}{(b-c)(b-a)}+\dfrac{2c-a-b}{(c-a)(c-b)}.$

22. $\dfrac{\dfrac{1}{x+1}-\dfrac{2}{(x+2)(x+1)}}{\dfrac{1}{x+2}-\dfrac{1}{(x+1)(x+2)}}.$

23. $\dfrac{\dfrac{1}{a^2}+\dfrac{1}{b^2}}{\dfrac{1}{a^2}-\dfrac{1}{b^2}}\div\dfrac{\dfrac{1}{a}+\dfrac{1}{b}}{\dfrac{1}{a}-\dfrac{1}{b}}.$

24. $\left(1-\dfrac{1-x}{1+x}+\dfrac{1+2x^2}{1-x^2}\right)\left(\dfrac{x+1}{2x+1}\right).$

25. $\dfrac{x^3-8y^3}{x(x-y)}\times\dfrac{x^2-xy+y^2}{x^2+2xy+4y^2}\times\dfrac{x(x^2-y^2)}{x^3+y^3}.$

26. $\dfrac{2}{a}+\dfrac{2}{b}+\dfrac{2}{c}-\dfrac{b+c-a}{bc}-\dfrac{c+a-b}{ac}-\dfrac{a+b-c}{ab}.$

CHAPTER X.

FRACTIONAL EQUATIONS.

Reduction of Equations Containing Fractions.

1. Solve $\frac{x}{3} - \frac{x-1}{11} = x - 9.$

Multiply by 33, the L. C. M. of the denominators.

Then,
$$11x - 3x + 3 = 33x - 297,$$
$$11x - 3x - 33x = -297 - 3,$$
$$-25x = -300.$$
$$\therefore x = 12.$$

Note. Since the minus sign precedes the second fraction, in removing the denominator the sign of every term of the numerator is changed.

2. Solve $\frac{2x+1}{2x-1} - \frac{2x-1}{2x+1} = \frac{8}{4x^2-1}.$

The L. C. D. $= (2x+1)(2x-1)$.

Multiply by the L. C. D., and we have,
$$4x^2 + 4x + 1 - (4x^2 - 4x + 1) = 8.$$
$$\therefore 4x^2 + 4x + 1 - 4x^2 + 4x - 1 = 8.$$
Reducing,
$$x = 1.$$

174. To Clear an Equation of Fractions, therefore,

Multiply each term by the L. C. M. of the denominators.

If a fraction is preceded by a **minus sign,** *the sign of every term of the numerator must be changed when the denominator is removed.*

Solve $$\frac{x-4}{x-5}-\frac{x-5}{x-6}=\frac{x-7}{x-8}-\frac{x-8}{x-9}.$$

NOTE. The solution of this and similar problems will be much easier by combining the fractions on the left side and the fractions on the right side than by the rule.

$$\frac{(x-4)(x-6)-(x-5)^2}{(x-5)(x-6)}=\frac{(x-7)(x-9)-(x-8)^2}{(x-8)(x-9)}.$$

By simplifying the numerators, we have

$$\frac{-1}{(x-5)(x-6)}=\frac{-1}{(x-8)(x-9)}.$$

Since the numerators are equal, the denominators are equal.

Hence, $(x-5)(x-6)=(x-8)(x-9)$

Solving, we have $x=7.$

EXERCISE 64.

Solve:

1. $\frac{3x-1}{4}-\frac{2x+1}{3}-\frac{4x-5}{5}=4.$

2. $2-\frac{7x-1}{6}=3x-\frac{19x+3}{4}.$

3. $\frac{5x+1}{3}+\frac{19x+7}{9}-\frac{3x-1}{2}=\frac{7x-1}{6}.$

4. $4+\frac{x}{7}=\frac{3x-2}{2}-\frac{11x+2}{14}-\frac{2-7x}{3}.$

5. $\frac{9x+5}{14}+\frac{8x-7}{14}=\frac{36x+15}{56}+\frac{41}{56}.$

6. $11-\left(\frac{3x-1}{4}+\frac{2x+1}{3}\right)=10-\left(\frac{2x-5}{3}+\frac{7x-1}{8}\right).$

7. $\frac{7x-4}{9}+\frac{3x-1}{5}-\frac{5(x-1)}{6}=\frac{3(3x-1)}{20}+\frac{x}{7}.$

8. $$6x - \frac{27x-1}{4} - \frac{2(4x-1)}{5} = \frac{9x-5}{4} - \frac{11x-2}{3} + 22.$$

9. $$\frac{10x+11}{6} - \frac{12x-13}{3} - 4 = \frac{7-6x}{4}.$$

10. $$\frac{3}{y-4} + \frac{5}{2(y-4)} + \frac{9}{2(y-4)} = \frac{1}{2}.$$

11. $$\frac{2}{x-1} - \frac{5}{2(x-1)} = \frac{8}{3(x-1)} - \frac{x}{x-1} + \frac{5}{18}.$$

12. $$\frac{2x-3}{2x-4} - 6 = \frac{x+5}{3x-6} - \frac{11}{2}.$$

13. $$\frac{10-7x}{6-7x} = \frac{5x-4}{5x}.$$

14. $$\frac{5+8x}{3+2x} = \frac{45-8x}{13-2x}.$$

15. $$\frac{5x-1}{2x+3} = \frac{5x-3}{2x-3}.$$

16. $$\frac{x}{3} - \frac{x^2-5x}{3x-7} = \frac{2}{3}.$$

17. $$\frac{2x+1}{2x-1} - \frac{8}{4x^2-1} = \frac{2x-1}{2x+1}.$$

18. $$\frac{5-2x}{x-1} - \frac{2-7x}{x+1} = \frac{5x^2+4}{x^2-1}.$$

19. $$\frac{6}{x+2} - \frac{x+2}{x-2} + \frac{x^2}{x^2-4} = 0.$$

20. $$\frac{4}{1+x} + \frac{x+1}{1-x} - \frac{x^2-3}{1-x^2} = 0.$$

21. $$\frac{2x+1}{3x-3} = \frac{7x-1}{6x+6} - \frac{2x^2-3x-45}{4x^2-4}.$$

22. $$\frac{x^2-x+1}{x-1} + \frac{x^2+x+1}{x+1} = 2x.$$

23. $$\frac{9x+5}{6(x-1)} + \frac{3x^2-51x-71}{18(x^2-1)} = \frac{15x-7}{9(x+1)}.$$

24. $$\frac{4}{x+2} + \frac{7}{x+3} - \frac{37}{x^2+5x+6} = 0.$$

25. $$\frac{1}{x-1} - \frac{1}{x-2} = \frac{1}{x-3} - \frac{1}{x-4}.$$

175. If the denominators contain both simple and compound expressions, it is best to remove the simple expressions first, and then each compound expression in turn. After each multiplication the result should be reduced to the simplest form.

1. Solve $\frac{8x+5}{14}+\frac{7x-3}{6x+2}=\frac{4x+6}{7}$.

Multiply both sides by 14,

Then, $$8x+5+\frac{7(7x-3)}{3x+1}=8x+12.$$

Transpose and combine, $$\frac{7(7x-3)}{3x+1}=7.$$

Divide by 7 and multiply by $3x+1$,

$$7x-3=3x+1.$$

$$\therefore x=1.$$

2. Solve $\dfrac{3-\dfrac{4x}{9}}{4}=\dfrac{1}{4}-\dfrac{\dfrac{7x}{9}-3}{10}$.

Simplify the complex fractions by multiplying both terms of each fraction by 9.

Then, $$\frac{27-4x}{36}=\frac{1}{4}-\frac{7x-27}{90}.$$

Multiply both sides by 180, the L. C. D.,

$$135-20x=45-14x+54,$$

$$-6x=-36.$$

$$\therefore x=6.$$

EXERCISE 65.

Solve:

1. $\frac{4x+3}{10}-\frac{2x-5}{5x-1}=\frac{2x-1}{5}$.

2. $\frac{9x+20}{36}=\frac{4x-12}{5x-4}+\frac{x}{4}$.

3. $\dfrac{10x+17}{18} - \dfrac{12x+2}{13x-16} = \dfrac{5x-4}{9}.$

4. $\dfrac{6x+7}{9} + \dfrac{7x-13}{6x+3} = \dfrac{2x+4}{3}.$

5. $\dfrac{6x+7}{15} - \dfrac{2x-2}{7x-6} = \dfrac{2x+1}{5}.$

6. $\dfrac{6x+1}{15} - \dfrac{2x-4}{7x-16} = \dfrac{2x-1}{5}.$

7. $\dfrac{11x-13}{14} - \dfrac{22x-75}{28} = \dfrac{13x+7}{2(3x+7)}.$

8. $\dfrac{2x+8\frac{1}{2}}{9} - \dfrac{13x-2}{17x-32} + \dfrac{x}{3} = \dfrac{7x}{12} - \dfrac{x+16}{36}.$

9. $\dfrac{6-5x}{15} - \dfrac{7-2x^2}{14(x-1)} = \dfrac{1+3x}{21} - \dfrac{2x-2\frac{1}{5}}{6} + \dfrac{1}{105}.$

10. $\dfrac{2x-5}{5} + \dfrac{x-3}{2x-15} = \dfrac{4x-3}{10} - 1\tfrac{1}{10}.$

Literal Equations.

176. Literal equations are equations in which some or all of the given numbers are represented by letters; known numbers are represented by the *first* letters of the alphabet.

Solve $(a-x)(a+x) = 2a^2 + 2ax - x^2$.

Then, $a^2 - x^2 = 2a^2 + 2ax - x^2,$

$-2ax = a^2.$

$\therefore x = -\dfrac{a}{2}.$

EXERCISE 66.

Solve:

1. $ax + 2b = 3bx + 4a.$
2. $x^2 + b^2 = (a-x)^2.$
3. $(a+x)(b+x) = x(x-c).$
4. $(x-a)(x+b) = (x-b)(x-c).$

5. $\dfrac{x}{a+ax}=\dfrac{b}{c+cx}.$

6. $\dfrac{c+d}{(ab+bx)}=\dfrac{m-x}{(an+nx)}.$

7. $\dfrac{x+2}{x-2}=\dfrac{m+n}{m-n}.$

8. $\dfrac{m+n}{2+x}=\dfrac{m-n}{2-x}.$

9. $\dfrac{a+bx}{a+b}=\dfrac{c+dx}{c+d}.$

10. $\dfrac{6x+a}{4x+b}=\dfrac{3x-b}{2x-a}.$

11. $\dfrac{x}{a}+\dfrac{x}{b}+\dfrac{x}{c}=d.$

12. $\dfrac{ax-b}{c}-\dfrac{bx+c}{a}=abc.$

13. $\dfrac{5ax}{a-b}-3a=8x.$

14. $\dfrac{x}{a-b}-\dfrac{5a}{a+b}=\dfrac{2bx}{a^2-b^2}.$

15. $\dfrac{1}{n}+\dfrac{n}{x+n}=\dfrac{x+n}{nx}.$

16. $\dfrac{x}{a}+\dfrac{x}{b-a}=\dfrac{a}{b+a}.$

17. $\dfrac{x-a}{x-b}=\left(\dfrac{2x-a}{2x-b}\right)^2.$

18. $\dfrac{x-a}{2}=\dfrac{(x-b)^2}{2x-a}.$

19. $\dfrac{\dfrac{m+n}{x}}{\dfrac{1}{m}}=\dfrac{a}{b}.$

20. $\dfrac{\dfrac{ax-b}{x}}{\dfrac{a}{b}}=\dfrac{\dfrac{ax+b}{x}}{\dfrac{b}{a}}.$

21. $\dfrac{\dfrac{cx+d}{a}}{\dfrac{cx}{d}}=\dfrac{2d}{a}.$

22. $\dfrac{\dfrac{a+x-3}{3}}{\dfrac{a-x+3}{3}}=\dfrac{b}{c}.$

23. $\dfrac{a-b}{bx+c}+\dfrac{a+b}{ax-c}=0.$

24. $\dfrac{x+a}{x-a}-\dfrac{x-b}{x+b}=\dfrac{2(a+b)}{x}.$

25. $\dfrac{x+a}{b}-x=b-\dfrac{x-b}{c}+\dfrac{c-bx}{b}.$

26. $\dfrac{20a-bx}{5a}+\dfrac{9c-ax}{3c}+\dfrac{6d-cx}{2d}=10.$

27. $\dfrac{ax}{b}-\dfrac{b-x}{2c}+\dfrac{a(b-x)}{3d}=a.$

Problems Involving Fractional Equations.

EXERCISE 67.

1. The sum of the third and fourth parts of a certain number exceeds 3 times the difference of the fifth and sixth parts by 29. Find the number.

Let $x =$ the number.

Then, $\frac{x}{3} + \frac{x}{4} =$ the sum of its third and fourth parts,

$\frac{x}{5} - \frac{x}{6} =$ the difference of its fifth and sixth parts,

$3\left(\frac{x}{5} - \frac{x}{6}\right) =$ 3 times the difference of its fifth and sixth parts,

$\frac{x}{3} + \frac{x}{4} - 3\left(\frac{x}{5} - \frac{x}{6}\right) =$ the given excess.

But $29 =$ the given excess.

$$\therefore \frac{x}{3} + \frac{x}{4} - 3\left(\frac{x}{5} - \frac{x}{6}\right) = 29.$$

Multiply by 60, the L. C. D. of the fractions.

$$20x + 15x - 36x + 30x = 60 \times 29.$$

Combining, $29x = 60 \times 29.$

Dividing by 29, $x = 60.$

The required number, therefore, is 60.

2. The difference between the fifth and ninth parts of a certain number is 4. Find the number.

3. One half of a certain number exceeds the sum of its fifth and ninth parts by 17. Find the number.

4. The sum of the third and sixth parts of a certain number exceeds the difference of its sixth and seventh parts by 20. Find the number.

5. There are two consecutive numbers, x and $x + 1$, such that one half of the larger exceeds one third of the smaller number by 9. Find the numbers.

6. The sum of two numbers is 63, and if the greater is divided by the smaller number, the quotient is 2 and the remainder 3. Find the numbers.

Let $x =$ the greater number.
Then $63 - x =$ the smaller number.

Since the quotient $= \dfrac{\text{Dividend} - \text{Remainder}}{\text{Divisor}}$,

and since, in this problem, the dividend is x, the remainder is 3, and the divisor is $63 - x$, we have

Chain

$$\frac{x-3}{63-x} = 2.$$

Solving, $x = 43$.

The two numbers, therefore, are 43 and 20.

7. The sum of two numbers is 120, and if the greater is divided by the smaller number, the quotient is 2 and the remainder 15. Find the numbers.

8. The sum of two numbers is 126, and if the greater is divided by the smaller number, the quotient is 3 and the remainder 10. Find the numbers.

9. The difference of two numbers is 51, and if the greater is divided by the smaller, the quotient is 4 and the remainder 6. Find the numbers.

10. The difference of two numbers is 87, and if the greater is divided by the smaller, the quotient is 8 and the remainder 10. Find the numbers.

11. Divide 450 into two parts such that the smaller part is contained in the larger part 9 times, with a remainder of 20.

12. The difference of two numbers is 25, and if the greater is divided by the smaller, the quotient is 4 and the remainder 4. Find the numbers.

13. Eight years ago a boy was one fourth as old as he will be one year hence. How old is he now?

Let x = the number of years old he is now.

Then $x - 8$ = the number of years old he was eight years ago,

and $x + 1$ = the number of years old he will be one year hence.

$$\therefore x - 8 = \tfrac{1}{4}(x + 1).$$

Solving, $x = 11.$

Therefore, the boy is 11 years old.

14. A son is one third as old as his father. In 10 years he will be one half as old. Find the age of the son.

15. B's age is one fifth of A's age. In 12 years B's age will be one third of A's age. Find their ages.

16. The sum of the ages of A and B is 60 years, and 10 years hence B's age will be one third of A's. Find their ages.

17. A father is 40 years old, and his son is one fourth of that age. In how many years will the son be half as old as his father?

18. A is 30 years old, and B's age is two thirds of A's. How many years ago was B's age one third of A's?

19. A son is one fourth as old as his father. Four years ago he was only one fifth as old as his father. What is the age of each?

20. A is 40 years old, and B is half as old as A. In how many years will B be two thirds as old as A?

21. B is one third as old as A. Ten years ago he was one fourth as old as A. What are their present ages?

22. The sum of the ages of a father and his son is 75 years. The son's age increased by 5 years is one fourth of the father's age. Find their ages.

23. A rectangle has its length 6 feet more and its width 5 feet less than the side of the equivalent square. Find the dimensions of the rectangle.

Let $x =$ the number of feet in a side of the square.
Then $x + 6 =$ the number of feet in the length of the rectangle,
and $x - 5 =$ the number of feet in the width of the rectangle.

Since the area of a rectangle is equal to the product of the number of units of length in the length and width of the rectangle,

$(x + 6)(x - 5) =$ the area of the rectangle in square feet,
and $x \times x =$ the area of the square in square feet.

But these areas are equal.

$$\therefore (x + 6)(x - 5) = x^2.$$

Solving, $x = 30.$

Therefore, the dimensions of the rectangle are 36 feet and 25 feet.

24. A rectangle has its length and breadth, respectively, 12 feet longer and 8 feet shorter than the side of the equivalent square. Find its area.

25. The length of a floor exceeds the breadth by 6 feet. If each dimension were 1 foot more, the area of the floor would be 41 sq. ft. more. Find its dimensions.

26. A rectangle whose length is 8 feet more than its breadth would have its area 35 sq. ft. more, if each dimension were 1 foot more. Find its dimensions.

27. The length of a rectangle exceeds its width by 4 feet. If the length were diminished by 2 feet and the width by 2 feet, the area would be diminished by 40 sq. ft. Find its dimensions.

28. The length of a floor exceeds its width by 8 feet. If each dimension were 2 feet more, the area would be 124 sq. ft. more. Find its dimensions.

29. A can do a piece of work in 2 days, and B can do it in 3 days. How many days will it take both together to do the work?

Let $x =$ the number of days it will take both together.

Then $\frac{1}{x} =$ *the part both together can do in one day,*

$\frac{1}{2} =$ the part A can do in one day,

$\frac{1}{3} =$ the part B can do in one day,

and $\frac{1}{2} + \frac{1}{3} =$ *the part both together can do in one day.*

$$\therefore \frac{1}{2} + \frac{1}{3} = \frac{1}{x}.$$

5 = 6x

-6x = -5

Solving, $x = 1\frac{1}{5}$.

Therefore, they together can do the work in $1\frac{1}{5}$ days.

30. A can do a piece of work in 3 days, B in 4 days, and C in 5 days. How many days will it take them to do it working together?

31. A can do a piece of work in 6 days, B in 5 days, and C in 4 days. How many days will it take them together to do the work?

32. A can do a piece of work in $2\frac{1}{2}$ days, B in $3\frac{1}{3}$ days, and C in $3\frac{3}{4}$ days. How many days will it take them together to do the work?

33. A can do a piece of work in 8 days, B in 10 days; A and B together, with the help of C, can do the work in 3 days. How many days will it take C alone to do the work?

34. A and B together can mow a field in 8 hours, A and C in 10 hours, and A alone in 15 hours. In how many hours can B and C together mow the field?

8 **35.** A and B together can build a wall in 12 days, A and C in 15 days, B and C in 20 days. In how many days can they build the wall if they all work together?

HINT. By working 2 *days each* they build $\frac{1}{12} + \frac{1}{15} + \frac{1}{20}$ of it. Hence, in *one* day they can build $\frac{1}{2}(\frac{1}{12} + \frac{1}{15} + \frac{1}{20})$.

36. A cistern can be filled by two pipes in 15 and 20 hours, respectively; and can be emptied by a waste pipe in 30 hours. In how many hours will it be filled if all the pipes together are open?

Let $x =$ the number of hours if all the pipes are open.

Then $\frac{1}{x} =$ the part filled in one hour if all the pipes are open.

$\frac{1}{15} + \frac{1}{20} - \frac{1}{30} =$ the part all together can fill in one hour.

$$\therefore \frac{1}{15} + \frac{1}{20} - \frac{1}{30} = \frac{1}{x}.$$

Solving, $x = 12.$

Therefore, if all the pipes are open it will be filled in 12 hours.

37. A cistern can be filled by three pipes in 8, 12, and 16 hours, respectively. In how many hours will it be filled by all the pipes together?

38. A cistern can be filled by two pipes in 4 hours and 5 hours, respectively, and can be emptied by a third pipe in 6 hours. In how many hours will the cistern be filled if the pipes are all running together?

39. A tank can be filled by three pipes in 1 hour and 40 minutes, 3 hours and 20 minutes, and 5 hours, respectively. In what time will the tank be filled if all three pipes are running together?

40. A cistern can be filled by three pipes in $2\frac{1}{3}$ hours, $3\frac{1}{2}$ hours, and $4\frac{2}{3}$ hours, respectively. In how many hours will the cistern be filled if all the pipes are running together?

41. A cistern has three pipes. The first pipe will fill the cistern in 6 hours, the second in 10 hours, and all three pipes together will fill it in 3 hours. How long will it take the third pipe alone to fill it?

42. A courier who travels 6 miles an hour is followed, after 2 hours, by a second courier who travels $7\frac{1}{2}$ miles an hour. In how many hours will the second courier overtake the first?

Let $x =$ the number of hours the first travels.
Then $x - 2 =$ the number of hours the second travels,
$6x =$ the number of miles the first travels,
and $(x - 2)\,7\frac{1}{2} =$ the number of miles the second travels.

They both travel the same distance.

$$\therefore 6x = (x - 2)\,7\tfrac{1}{2},$$

or

$$12x = 15x - 30.$$

$$\therefore x = 10.$$

Therefore, the second courier will overtake the first in $10 - 2$, or 8 hours.

43. A sets out from Boston and walks towards Portland at the rate of 3 miles an hour. Three hours afterward B sets out from the same place and walks in the same direction at the rate of $3\frac{1}{2}$ miles an hour. How far from Boston will B overtake A?

44. A courier who goes at the rate of $4\frac{1}{2}$ miles an hour is followed, after 4 hours, by another who goes at the rate of $5\frac{1}{2}$ miles an hour. In how many hours will the second overtake the first?

45. A person walks to the top of a mountain at the rate of $1\frac{1}{2}$ miles an hour, and down the same way at the rate of $4\frac{1}{2}$ miles an hour. If he is out 6 hours, how far is it to the top of the mountain?

46. In going a certain distance, a train traveling at the rate of 50 miles an hour takes 2 hours less than a train traveling 40 miles an hour. Find the distance.

47. Find the time between 2 and 3 o'clock when the hands of a clock are together.

At 2 o'clock the hour-hand is 10 minute-spaces ahead of the minute-hand.

Let $x =$ *the number of spaces the minute-hand moves over.*

Then $x - 10 =$ the number of spaces the hour-hand moves over.

Now, as the minute-hand moves 12 times as fast as the hour-hand,

$12(x - 10) =$ *the number of spaces the minute-hand moves over.*

$$\therefore 12(x - 10) = x,$$

and
$$11x = 120.$$

$$\therefore x = 10\tfrac{10}{11}.$$

Therefore, the time is $10\frac{10}{11}$ minutes past 2 o'clock.

48. Find the time between 4 and 5 o'clock when the hands of a clock are together. 21 9/11

49. Find the time between 3 and 4 o'clock when the hands of a clock are at right angles to each other. 37 [illegible]

HINT. In this case the minute-hand is 15 minutes ahead of the hour-hand.

16 1/2 [illegible]

50. Find the time between 2 and 3 o'clock when the hands of a clock point in opposite directions.

HINT. In this case the minute-hand is 30 minutes ahead of the hour-hand, or 30 minutes behind it.

6 past 4, or

51. Find the times between 4 and 5 o'clock when the hands of a clock are at right angles to each other. 41 [illegible]

52. Find the time between 1 and 2 o'clock when the hands of a clock point in opposite directions. 41 [illegible]

53. At what time between 6 and 7 o'clock are the hands of a watch together? 37 1/2 w[illegible]

54. A hare takes 4 leaps to a greyhound's 3; but 2 of the greyhound's leaps are equivalent to 3 of the hare's. The hare has a start of 50 of her own leaps. How many leaps must the greyhound take to catch the hare?

Let $3x =$ the number of leaps taken by the greyhound.
Then $4x =$ the number of leaps of the hare in the same time.
Also, let $a =$ the number of feet in one leap of the hare.
Then $\frac{3a}{2} =$ the number of feet in one leap of the hound.
Therefore, $3x \times \frac{3a}{2}$ or $\frac{9ax}{2} =$ the whole distance.

As the hare has a start of 5) leaps, and takes $4x$ leaps more before she is caught, and as each leap is a feet,

$$(50 + 4x)a = \text{the whole distance.}$$

$$\therefore \frac{9ax}{2} = (50 + 4x)a.$$

Multiply by 2, $$9ax = (100 + 8x)a.$$

Divide by a, $$9x = 100 + 8x,$$

$$x = 100,$$

$$\therefore 3x = 300.$$

Therefore, the greyhound must take 300 leaps.

55. A hound takes 3 leaps while a rabbit takes 5; but 1 of the hound's leaps is equivalent to 2 of the rabbit's. The rabbit has a start of 120 of her own leaps. How many leaps will the rabbit take before she is caught?

56. A rabbit takes 6 leaps to a dog's 5, and 7 of the dog's leaps are equivalent to 9 of the rabbit's. The rabbit has a start of 60 of her own leaps. How many leaps must the dog take to catch the rabbit?

57. A dog takes 4 leaps while a rabbit takes 5; but 3 of the dog's leaps are equivalent to 4 of the rabbit's. The rabbit has a start of 90 of the *dog's leaps*. How many leaps will each take before the rabbit is caught?

58. A merchant adds yearly to his capital one third of it, but takes from it, at the end of each year, \$5000 for expenses. At the end of the third year, after deducting the last \$5000, he has twice his original capital. How much had he at first?

Let $x =$ number of dollars he had at first.

Then $\frac{4x}{3} - 5000$, or $\frac{4x - 15000}{3}$,

will stand for the number of dollars at the end of first year,

and $\frac{4}{3}\left(\frac{4x - 15000}{3}\right) - 5000$, or $\frac{16x - 105000}{9}$,

will stand for the number of dollars at the end of second year,

and $\frac{4}{3}\left(\frac{16x - 105000}{9}\right) - 5000$, or $\frac{64x - 555000}{27}$,

will stand for the number of dollars at the end of third year.

But $2x$ stands for the number of dollars at the end of third year.

$$\therefore \frac{64x - 555000}{27} = 2x.$$

Whence $x = 55,500$.

Therefore, the merchant had \$55,500 at first.

59. A trader adds yearly to his capital one fourth of it, but takes from it, at the end of each year, \$800 for expenses. At the end of the third year, after deducting the last \$800, he has $1\frac{7}{8}$ of his original capital. How much had he at first?

60. A trader adds yearly to his capital one fifth of it, but takes from it, at the end of each year, \$2500 for expenses. At the end of the third year, after deducting the last \$2500, he has $1\frac{7}{10}$ of his original capital. Find his original capital.

61. A trader maintained himself for three years at an expense of \$500 a year; and each year increased that part of his stock which was not so expended by one third of it. At the end of the third year his original stock was doubled. What was his original stock?

62. The sum of the third, fourth, and fifth parts of a number exceeds the half of the number by 17. Find the number.

63. There are two consecutive numbers, x and $x+1$, such that one fourth of the smaller exceeds one ninth of the larger by 11. Find the numbers.

f 64. Find three consecutive numbers such that if they are divided by 7, 10, and 17, respectively, the sum of the quotients will be 15.

65. In a mixture of alcohol and water the alcohol was 24 gallons more than half the mixture, and the water was 4 gallons less than one fourth the mixture. How many gallons were there of each?

66. The width of a room is three fourths of its length. If the width was 4 feet more and the length 4 feet less, the room would be square. Find its dimensions.

67. The difference of two numbers is 40, and if the greater is divided by the less the quotient is 4, and the remainder 4. Find the numbers.

68. Divide the number 240 into two parts such that the smaller part is contained in the larger part 5 times, with a remainder of 6.

69. A can do a piece of work in 3 days, B in 4 days, and C in 6 days. How many days will it take them to do it working together?

70. At what time between 2 and 3 o'clock are the hands of a watch at right angles?

71. Find a number such that the sum of its sixth and ninth parts shall exceed the difference of its ninth and twelfth parts by 9.

72. The sum of two numbers is 91, and if the greater is divided by the less the quotient is 2, and the remainder is 1. Find the numbers.

73. A is 60 years old, and B is three fourths as old. How many years since B was just half as old as A?

74. A can do a piece of work in $2\frac{1}{3}$ days, B in $3\frac{1}{2}$ days, and C in $4\frac{2}{3}$ days. How long will it take them to do it working together?

75. A and B can separately do a piece of work in 12 days and 20 days, and with the help of C they can do it in 6 days. How long would it take C alone to do the work?

76. A rectangle has its length 4 feet longer and its width 3 feet shorter than the side of the equivalent square. Find its area.

77. A person has 6 hours at his disposal. How far may he ride at 6 miles an hour so as to return in time, walking back at the rate of 3 miles an hour?

78. A boy starts from Exeter and walks towards Andover at the rate of 3 miles an hour, and 2 hours later another boy starts from Andover and walks towards Exeter at the rate of $2\frac{1}{2}$ miles an hour. The distance from Exeter to Andover is 28 miles. How far from Exeter will they meet?

79. A dog makes 4 leaps while a hare makes 5, but 3 of the dog's leaps are equal to 4 of the hare's. The hare has a start of 60 of the dog's leaps. How many leaps will each make before the hare is caught?

80. At what time between 3 and 4 o'clock are the hands of a watch pointing in opposite directions?

81. In going from Boston to Portland, a passenger train, at 36 miles an hour, occupies 1 hour less time than a freight train at 27 miles an hour. Find the distance from Boston to Portland.

82. A cistern can be filled by three pipes in 15, 20, and 30 minutes, respectively. In what time will it be filled if the pipes are all running together?

83. A cistern can be filled by two pipes in 25 minutes and 30 minutes, respectively, and emptied by a third in 20 minutes. In what time will it be filled if all three pipes are running together?

84. A hare takes 7 leaps while a dog takes 5, and 5 of the dog's leaps are equal to 8 of the hare's. The hare has a start of 50 of her own leaps. How many leaps will the hare take before she is caught?

85. The width of a rectangle is an inch more than half its length, and if a strip 5 inches wide is taken off from the four sides, the area of the strip is 510 square inches. Find the dimensions of the rectangle.

86. A and B together can do a piece of work in 10 days, A and C in 12 days, and A by himself in 18 days. How many days will it take B and C together to do the work? How many days will it take A, B, and C together?

87. A and B can do a piece of work in 10 days, A and C in 12 days, B and C in 15 days. How long will it take them to do the work if they all work together?

88. A sets out and travels at the rate of 9 miles in 2 hours. Seven hours afterwards B sets out from the same place and travels in the same direction at the rate of 5 miles an hour. In how many hours will B overtake A?

89. A man walks to the top of a mountain at the rate of $2\frac{1}{2}$ miles an hour, and down the same way at the rate of 4 miles an hour, and is out 5 hours. How far is it to the top of the mountain?

90. A tank can be filled by three pipes in 1 hour 20 minutes, 2 hours 20 minutes, and 3 hours 20 minutes, respectively. In how many minutes can it be filled by all three together?

91. A's age now is two fifths of B's. Eight years ago A's age was two ninths of B's. Find their ages.

92. A had five times as much money as B. He gave B 5 dollars, and then had only twice as much as B. How much had each at first?

93. At what time between 12 and 1 o'clock are the hour and minute-hands pointing in opposite directions?

94. Eleven sixteenths of a certain principal was at interest at 5 per cent, and the balance at 4 per cent. The entire income was $1500. Find the principal.

95. A train that travels 36 miles an hour is $\frac{3}{4}$ of an hour in advance of a second train that travels 42 miles an hour. In how long a time will the last train overtake the first?

96. An express train that travels 40 miles an hour starts from a certain place 50 minutes after a freight train, and overtakes the freight train in 2 hours 5 minutes. Find the rate per hour of the freight train.

97. If 1 pound of tin loses $\frac{5}{37}$ of a pound, and 1 pound of lead loses $\frac{2}{23}$ of a pound, when weighed in water, how many pounds of tin and of lead in a mass of 60 pounds that loses 7 pounds when weighed in water?

98. If 19 ounces of gold lose 1 ounce, and 10 ounces of silver lose 1 ounce, when weighed in water, how many ounces of gold and of silver in a mass of gold and silver that weighs 530 ounces in air and 495 ounces in water?

99. A messenger starts to carry a despatch, and 5 hours later a second messenger sets out to overtake the first in 8 hours. In order to do this, he is obliged to travel $2\frac{1}{2}$ miles an hour more than the first. How many miles an hour does the first travel?

100. The fore and hind wheels of a carriage are respectively $9\frac{1}{2}$ feet and $11\frac{2}{5}$ feet in circumference. What distance will the carriage have made when one of the fore wheels has made 160 revolutions more than one of the hind wheels?

101. When a certain brigade of troops is formed in a solid square there is found to be 100 men over; but when formed in column with 5 men more in front and 3 men less in depth than before, the column needs 5 men to complete it. Find the number of troops.

102. An officer can form his men in a hollow square 14 deep. The whole number of men is 3136. Find the number of men in the front of the hollow square.

103. A trader increases his capital each year by one fourth of it, and at the end of each year takes out $2400 for expenses. At the end of 3 years, after deducting the last $2400, he finds his capital to be $10,000. Find his original capital.

104. A and B together can do a piece of work in $1\frac{1}{2}$ days, A and C together in $1\frac{3}{4}$ days, and B and C together in $1\frac{7}{8}$ days. How many days will it take each alone to do the work?

105. A fox pursued by a hound has a start of 100 of her leaps. The fox makes 3 leaps while the hound makes 2; but 3 leaps of the hound are equivalent to 5 of the fox. How many leaps will each take before the hound catches the fox?

Formulas and Rules.

177. When the given numbers of a problem are represented by letters, the result obtained from solving the problem is a general expression which includes all problems of that class. Such an expression is called a **formula,** and the translation of this formula into words is called a **rule.**

We will illustrate by examples:

1. The sum of two numbers is s, and their difference d; find the numbers.

Let $x =$ the smaller number;

then, $x + d =$ the larger number.

Hence, $x + x + d = s,$

or $2x = s - d.$

$$\therefore x = \frac{s-d}{2},$$

$$x + d = \frac{s - d + 2d}{2},$$

$$= \frac{s+d}{2}.$$

Therefore, the numbers are $\frac{s+d}{2}$ and $\frac{s-d}{2}$.

As these formulas hold true whatever numbers s and d stand for, we have the general rule for finding two numbers when their sum and difference are given:

Add the difference to the sum and take half the result for the greater number.

Subtract the difference from the sum and take half the result for the smaller number.

2. If A can do a piece of work in a days, and B can do the same work in b days, in how many days can both together do it?

Let $x =$ the required number of days.

Then, $\frac{1}{x} =$ the part both together can do in one day.

Now, $\frac{1}{a} =$ the part A can do in one day,

and $\frac{1}{b} =$ the part B can do in one day;

therefore, $\frac{1}{a} + \frac{1}{b} =$ the part both together can do in one day.

$$\therefore \frac{1}{a} + \frac{1}{b} = \frac{1}{x}.$$

Whence,
$$x = \frac{ab}{a+b}.$$

The translation of this formula gives the following rule for finding the time required by two agents together to produce a given result, when the time required by each agent separately is known:

Divide the product of the numbers that express the units of time required by each to do the work by the sum of these numbers; the quotient is the time required by both together.

EXERCISE 68.

1. A person has a hours at his disposal. How far may he ride in a coach that travels b miles an hour, so as to return home in time, walking back at the rate of c miles an hour?

2 A courier who travels a miles a day is followed after c days by another who travels b miles a day. In how many days will the second overtake the first?

3. A has a dollars and B has b dollars. B gives A a certain number of dollars, and then has c times as much as A. How many dollars does A receive from B?

4. The fore wheel of a carriage is a feet in circumference, and the hind wheel is b feet. Find the distance traveled when the fore wheel has made c revolutions more than the hind wheel.

5. Two towns, P and Q, are a miles apart. One person sets out from P and travels towards Q at the rate of b miles an hour, and at the same time another person sets out from Q and travels towards P at the rate of c miles an hour. How many miles from P will they meet?

6. A person was employed a days on these conditions: for each day he worked he was to receive b cents, and for each day he was idle he was to forfeit c cents. At the end of a days he received d cents. How many days did he work?

7. A banker has two kinds of coins: it takes a pieces of the first to make a dollar, and b pieces of the second to make a dollar. A person wishes to obtain c pieces for a dollar. How many pieces of each kind must the banker give him?

Interest Formulas.

178. The elements involved in computation of interest are the *principal*, *rate*, *time*, *interest*, and *amount*.

Let $p =$ the principal,
$r =$ the interest of $1 for 1 year, at the given rate,
$t =$ the time expressed in years,
$i =$ the interest for the given time and rate,
$a =$ the amount (sum of principal and interest).

179. Given the Principal, Rate, and Time; to Find the Interest.

Since r is the interest of $1 for 1 year, pr is the interest of $\$p$ for 1 year, and prt is the interest of $\$p$ for t years.

$$\therefore \mathbf{i = prt}. \qquad \text{(Formula 1)}$$

RULE. *Find the product of the principal, rate, and time.*

180. Given the Interest, Rate, and Time; to Find the Principal.

By formula 1, $prt = i.$

Divide by rt, $\mathbf{p} = \dfrac{\mathbf{i}}{\mathbf{rt}}.$ (Formula 2)

181. Given the Amount, Rate, and Time; to Find the Principal.

From formula 1, $p + prt = a,$

or $p(1 + rt) = a.$

Divide by $1 + rt$, $\mathbf{p} = \dfrac{\mathbf{a}}{\mathbf{1 + rt}}.$ (Formula 3)

182. Given the Amount, Principal, and Rate; to Find the Time.

From formula 1, $p + prt = a.$

Transpose p, $prt = a - p.$

Divide by pr, $\mathbf{t} = \dfrac{\mathbf{a - p}}{\mathbf{pr}}.$ (Formula 4)

183. Given the Amount, Principal, and Time; to Find the Rate.

From formula 1, $p + prt = a.$

Transpose p, $prt = a - p.$

Divide by pt, $\mathbf{r} = \dfrac{\mathbf{a - p}}{\mathbf{pt}}.$ (Formula 5)

EXERCISE 69.

Solve by the preceding formulas:

1. The sum of two numbers is 40, and their difference is 10. Find the numbers.

2. The sum of two angles is 100°, and their difference is 21° 30′. Find the angles.

3. The sum of two angles is 116° 24′ 30″, and their difference is 56° 21′ 44″. Find the angles.

4. A can do a piece of work in 6 days, and B in 5 days. How long will it take both together to do it?

5. Find the interest of \$2750 for 3 years at $4\frac{1}{2}$ per cent.

6. Find the interest of \$950 for 2 years 6 months at 5 per cent.

7. Find the amount of \$2000 for 7 years 4 months at 6 per cent.

8. Find the rate if the interest on \$680 for 7 months is \$35.70.

9. Find the rate if the amount of \$750 for 4 years is \$900.

10. Find the rate if a sum of money doubles in 16 years 8 months.

11. Find the time required for the interest on \$2130 to be \$436.65 at 6 per cent.

12. Find the time required for the interest at 5 per cent on a sum of money to be equal to the principal.

13. Find the principal that will produce \$161.25 interest in 3 years 9 months at 8 per cent.

14. Find the principal that will amount to \$1500 in 3 years 4 months at 6 per cent.

15. How much money is required to yield \$2000 interest annually if the money is invested at 5 per cent?

16. Find the time in which \$640 will amount to \$1000 at 6 per cent.

17. Find the principal that will produce \$100 per month, at 6 per cent.

18. Find the rate if the interest on \$700 for 10 months is \$25.

CHAPTER XI.

SIMULTANEOUS SIMPLE EQUATIONS.

184. If we have *two* unknown numbers and but *one relation* between them, we can find an *unlimited number of pairs of values* for which the given relation will hold true.

Thus, if x and y are unknown, and we have given only the one relation $x + y = 10$, we can *assume* any value for x, and then from the relation $x + y = 10$ find the corresponding value of y. For from $x + y = 10$ we find $y = 10 - x$. If x stands for 1, y stands for 9; if x stands for 2, y stands for 8; if x stands for -2, y stands for 12; and so on without end.

185. We may, however, have two equations that express *different* relations between the two unknowns. Such equations are called **independent equations.**

Thus, $x + y = 10$ and $x - y = 2$ are independent equations, for they evidently express *different* relations between x and y.

186. Independent equations involving the *same* unknowns are called **simultaneous equations.**

If we have *two* unknowns, and have given *two* independent equations involving them, there is but *one* pair of values which will hold true for both equations.

Thus, if in § 184, besides the relation $x + y = 10$, we have also the relation $x - y = 2$, the only pair of values for which both equations will hold true is the pair $x = 6$, $y = 4$.

Observe that in this problem x stands for the same number in *both* equations; so also does y.

187. Simultaneous equations are solved by combining the equations so as to obtain a single equation with one unknown number; this process is called **elimination.**

There are three methods of elimination in general use:

I. By Addition or Subtraction.
II. By Substitution.
III. By Comparison.

188. Elimination by Addition or Subtraction.

1. Solve:

$$\left.\begin{aligned}5x-3y&=20\\2x+5y&=39\end{aligned}\right\}\qquad\begin{matrix}(1)\\(2)\end{matrix}$$

Multiply (1) by 5, and (2) by 3,

$$\begin{aligned}25x-15y&=100 &&(3)\\6x+15y&=117 &&(4)\end{aligned}$$

Add (3) and (4),

$$\begin{aligned}31x&=217\\\therefore x&=7.\end{aligned}$$

Substitute the value of x in (2),

$$\begin{aligned}14+5y&=39.\\\therefore y&=5.\end{aligned}$$

In this solution y is eliminated by *addition.*

2. Solve:

$$\left.\begin{aligned}6x+35y&=177\\8x-21y&=\ \ 33\end{aligned}\right\}\qquad\begin{matrix}(1)\\(2)\end{matrix}$$

Multiply (1) by 4, and (2) by 3,

$$\begin{aligned}24x+140y&=708 &&(3)\\24x-\ \ 63y&=\ \ 99 &&(4)\end{aligned}$$

Subtract,

$$\begin{aligned}203y&=609\\\therefore y&=3.\end{aligned}$$

Substitute the value of y in (2),

$$\begin{aligned}8x-63&=33.\\\therefore x&=12.\end{aligned}$$

In this solution x is eliminated by *subtraction.*

189. To Eliminate by Addition or Subtraction, therefore,

Multiply the equations by such numbers as will make the coefficients of one of the unknown numbers equal in the resulting equations.

Add the resulting equations, or subtract one from the other, according as these equal coefficients have unlike or like signs.

NOTE. It is generally best to select the letter to be eliminated that requires the smallest multipliers to make its coefficients equal; and the smallest multiplier for each equation is found by dividing the L. C. M. of the coefficients of this letter by the given coefficient in that equation. Thus, in example 2, the L. C. M. of 6 and 8 (the coefficients of x) is 24, and hence the smallest multipliers of the two equations are 4 and 3, respectively.

Sometimes the solution is simplified by first adding the given equations or by subtracting one from the other.

Solve:
$$\left.\begin{aligned} x + 49y &= 51 \\ 49x + y &= 99 \end{aligned}\right\} \qquad \begin{aligned} &(1) \\ &(2) \end{aligned}$$

Add (1) and (2), $50x + 50y = 150.$ (3)

Divide (3) by 50, $x + y = 3.$ (4)

Subtract (4) from (1), $48y = 48.$

$\therefore y = 1.$

Subtract (4) from (2), $48x = 96.$

$\therefore x = 2.$

EXERCISE 70.

Solve by addition or subtraction:

1. $\left.\begin{aligned} 5x + 2y &= 39 \\ 2x - y &= 3 \end{aligned}\right\}$

2. $\left.\begin{aligned} x + 3y &= 22 \\ 2x - 4y &= 4 \end{aligned}\right\}$

3. $\left.\begin{aligned} 7x - 2y &= 11 \\ x + 5y &= 28 \end{aligned}\right\}$

4. $\left.\begin{aligned} 4x - 5y &= 26 \\ 3x - 6y &= 15 \end{aligned}\right\}$

5. $\left.\begin{aligned} x + 2y &= 35 \\ 3x - 2y &= 17 \end{aligned}\right\}$

6. $\left.\begin{aligned} x + 4y &= 35 \\ 2x - 3y &= 26 \end{aligned}\right\}$

7. $\left.\begin{aligned}3x+5y&=50\\x-7y&=8\end{aligned}\right\}$

8. $\left.\begin{aligned}5x+2y&=36\\2x+3y&=43\end{aligned}\right\}$

9. $\left.\begin{aligned}3x+7y&=50\\5x-2y&=15\end{aligned}\right\}$

10. $\left.\begin{aligned}2x+y&=3\\7x+5y&=21\end{aligned}\right\}$

11. $\left.\begin{aligned}x+2y&=9\\3x-3y&=90\end{aligned}\right\}$

12. $\left.\begin{aligned}4x-3y&=39\\3x-4y&=17\end{aligned}\right\}$

13. $\left.\begin{aligned}7x-2y&=69\\x-10y&=39\end{aligned}\right\}$

14. $\left.\begin{aligned}3x+7y&=16\\2x+5y&=13\end{aligned}\right\}$

190. Elimination by Substitution.

Solve:

$$\left.\begin{aligned}5x+4y&=32\\4x+3y&=25\end{aligned}\right\}\qquad\begin{aligned}&(1)\\&(2)\end{aligned}$$

Transpose $4y$ in (1), $$5x=32-4y. \tag{3}$$

Divide by coefficient of x, $$x=\frac{32-4y}{5}. \tag{4}$$

Substitute the value of x in (2),

$$4\left(\frac{32-4y}{5}\right)+3y=25,$$

$$\frac{128-16y}{5}+3y=25,$$

$$128-16y+15y=125,$$

$$-y=-3.$$

$$\therefore y=3.$$

Substitute the value of y in (2),

$$4x+9=25.$$

$$\therefore x=4.$$

To Eliminate by Substitution, therefore,

From one of the equations obtain the value of one of the unknown numbers in terms of the other.

Substitute for this unknown number its value in the other equation, and reduce the resulting equation.

EXERCISE 71.

Solve by substitution:

1. $\left.\begin{aligned} 2x - 7y &= 0 \\ 3x - 5y &= 11 \end{aligned}\right\}$

2. $\left.\begin{aligned} 4x - 5y &= 4 \\ 3x - 2y &= 10 \end{aligned}\right\}$

3. $\left.\begin{aligned} 2x - 3y &= 1 \\ 3x - 2y &= 29 \end{aligned}\right\}$

4. $\left.\begin{aligned} x + y &= 19 \\ 2x + 7y &= 88 \end{aligned}\right\}$

5. $\left.\begin{aligned} 2x - y &= 5 \\ x + 2y &= 25 \end{aligned}\right\}$

6. $\left.\begin{aligned} 19x - 15y &= 23 \\ 13x - 5y &= 21 \end{aligned}\right\}$

7. $\left.\begin{aligned} x + 10y &= 73 \\ 7x - 2y &= 7 \end{aligned}\right\}$

8. $\left.\begin{aligned} 3x - 2y &= 28 \\ 2x + 5y &= 63 \end{aligned}\right\}$

9. $\left.\begin{aligned} 2x - 3y &= 23 \\ 5x + 2y &= 29 \end{aligned}\right\}$

10. $\left.\begin{aligned} 6x - 7y &= 11 \\ 5x - 6y &= 8 \end{aligned}\right\}$

11. $\left.\begin{aligned} 7x + 6y &= 20 \\ 2x + 5y &= 32 \end{aligned}\right\}$

12. $\left.\begin{aligned} x + 5y &= 37 \\ 3x + 2y &= 46 \end{aligned}\right\}$

13. $\left.\begin{aligned} 3x - 7y &= 40 \\ 4x - 3y &= 9 \end{aligned}\right\}$

14. $\left.\begin{aligned} 5x + 9y &= -17 \\ 3x + 11y &= 1 \end{aligned}\right\}$

191. Elimination by Comparison.

Solve:

$$\left.\begin{aligned} 2x - 5y &= 66 \qquad (1) \\ 3x + 2y &= 23 \qquad (2) \end{aligned}\right\}$$

Transpose $5y$ in (1) and $2y$ in (2),

$$2x = 66 + 5y, \tag{3}$$

$$3x = 23 - 2y. \tag{4}$$

Divide (3) by 2,

$$x = \frac{66 + 5y}{2}. \tag{5}$$

Divide (4) by 3,

$$x = \frac{23 - 2y}{3}. \tag{6}$$

Equate the values of x,

$$\frac{66 + 5y}{2} = \frac{23 - 2y}{3}. \tag{7}$$

Reduce (7),

$$\begin{aligned} 198 + 15y &= 46 - 4y,\\ 19y &= -152.\\ \therefore y &= -8. \end{aligned}$$

Substitute the value of y in (1),

$$\begin{aligned} 2x + 40 &= 66.\\ \therefore x &= 13. \end{aligned}$$

192. To Eliminate by Comparison, therefore,

From each equation obtain the value of one of the unknown numbers in terms of the other.

Form an equation from these equal values and reduce the equation.

EXERCISE 72.

Solve by comparison:

1. $\left.\begin{aligned} x + y &= 30\\ 3x - 2y &= 25 \end{aligned}\right\}$

2. $\left.\begin{aligned} 7x + 3y &= 70\\ 5x - 4y &= 7 \end{aligned}\right\}$

3. $\left.\begin{aligned} 9x + 4y &= 54\\ 4x + 9y &= 89 \end{aligned}\right\}$

4. $\left.\begin{aligned} 7x + 2y &= 63\\ 8x - y &= 3 \end{aligned}\right\}$

5. $\left.\begin{aligned} 2x - 33y &= 29\\ 3x - 47y &= 46 \end{aligned}\right\}$

6. $\left.\begin{aligned} 2x - y &= 9\\ 5x - 3y &= 14 \end{aligned}\right\}$

7. $\left.\begin{aligned} 11x - 7y &= 6\\ 9x - 5y &= 10 \end{aligned}\right\}$

8. $\left.\begin{aligned} 5x + 9y &= 188\\ 13x - 2y &= 57 \end{aligned}\right\}$

9. $\left.\begin{aligned} 2x - 3y &= 1\\ 5x + 2y &= 126 \end{aligned}\right\}$

10. $\left.\begin{aligned} 50x - 9y &= 1\\ 7x - y &= 3 \end{aligned}\right\}$

11. $\left.\begin{aligned} x + 21y &= 2\\ 27y + 2x &= 19 \end{aligned}\right\}$

12. $\left.\begin{aligned} 10x + 3y &= 174\\ 3x + 10y &= 125 \end{aligned}\right\}$

13. $\left.\begin{aligned} 6x - 13y &= 2\\ 5x - 12y &= 4 \end{aligned}\right\}$

14. $\left.\begin{aligned} 2x + y &= 108\\ 10x + 2y &= 60 \end{aligned}\right\}$

15. $\left.\begin{aligned} 3x - 5y &= 5\\ 7x + y &= 265 \end{aligned}\right\}$

16. $\left.\begin{aligned} 12x + 7y &= 176\\ 3y - 19x &= 3 \end{aligned}\right\}$

193. Each equation must be simplified, if necessary, before the elimination.

Solve:
$$\left.\begin{aligned}\tfrac{3}{4}x-\tfrac{1}{2}(y+1)&=1\\ \tfrac{1}{3}(x+1)+\tfrac{3}{4}(y-1)&=9\end{aligned}\right\}\qquad\begin{aligned}(1)\\(2)\end{aligned}$$

Multiply (1) by 4, and (2) by 12,
$$3x-2y-2=4, \qquad (3)$$
$$4x+4+9y-9=108. \qquad (4)$$

From (3), $$3x-2y=6. \qquad (5)$$

From (4), $$4x+9y=113. \qquad (6)$$

Multiply (5) by 4, and (6) by 3,
$$12x-\ \ 8y=\ \ 24 \qquad (7)$$
$$\underline{12x+27y=339} \qquad (8)$$

Subtract (7) from (8), $$35y=315$$
$$\therefore y=9.$$

Substitute value of y in (1), $$x=8.$$

EXERCISE 73.

Solve:

1. $$\left.\begin{aligned}\frac{x}{3}+\frac{y}{2}&=\frac{4}{3}\\ \frac{x}{2}+\frac{y}{3}&=\frac{7}{6}\end{aligned}\right\}$$

2. $$\left.\begin{aligned}\frac{x+y}{3}+\frac{y-x}{2}&=9\\ \frac{x}{2}+\frac{x+y}{9}&=5\end{aligned}\right\}$$

3. $$\left.\begin{aligned}\frac{4x+5y}{40}&=x-y\\ \frac{2x-y}{3}+2y&=\frac{1}{2}\end{aligned}\right\}$$

4. $$\left.\begin{aligned}\frac{x-1}{8}+\frac{y-2}{5}&=2\\ 2x+\frac{2y-5}{3}&=21\end{aligned}\right\}$$

5. $$\left.\begin{aligned}\frac{3x-5y}{2}+3&=\frac{2x+y}{5}\\ 8-\frac{x-2y}{4}&=\frac{x}{2}+\frac{y}{3}\end{aligned}\right\}$$

6. $$\left.\begin{aligned}\frac{x+3y}{x-y}&=8\\ \frac{7x-13}{3y-5}&=4\end{aligned}\right\}$$

7. $$\left.\begin{aligned}\frac{x+1}{3}-\frac{y+2}{4}&=\frac{2(x-y)}{5}\\ \frac{x+6}{4}-\frac{y+1}{3}&=\frac{2x-y}{2}\end{aligned}\right\}$$

8. $$\left.\begin{aligned}\frac{x+2y+1}{2x-y+1}&=2\\ \frac{3x-y+1}{x-y+3}&=5\end{aligned}\right\}$$

9. $$\left.\begin{aligned}\frac{x-2}{5}-\frac{10-x}{3}&=\frac{y-10}{4}\\ \frac{2y+4}{3}&=\frac{4x+y+26}{8}\end{aligned}\right\}$$

10. $$\left.\begin{aligned}\frac{2x-y+3}{3}&=\frac{x-2y+19}{4}\\ \frac{3x-4y+3}{4}&=\frac{2y-4x+21}{3}\end{aligned}\right\}$$

11. $$\left.\begin{aligned}\frac{7+x}{5}-\frac{2x-y}{4}&=3y-5\\ \frac{5y-7}{2}+\frac{4x-3}{6}&=18-5x\end{aligned}\right\}$$

12. $$\left.\begin{aligned}\frac{26+5x-6y}{13}&=4y-3x\\ 12+\frac{5x-6y}{6}&=\frac{3(x+2y)}{4}\end{aligned}\right\}$$

13. $$\left.\begin{aligned}\frac{x+8}{2}&=2-\frac{3y-x}{6}\\ \frac{2x+y}{2}-\frac{9x-7}{8}&=\frac{7y-4x+36}{16}\end{aligned}\right\}$$

14. $$\left.\begin{aligned}\frac{x+2y+3}{13}&=\frac{5y-4x-6}{3}\\ \frac{6x-5y+4}{3}&=\frac{3x+2y+1}{19}\end{aligned}\right\}$$

15. $$\left.\begin{aligned}\frac{x+y}{y-x}&=\frac{5}{3}\\ 3x-\frac{3y+44}{7}&=13\end{aligned}\right\}$$

16. $$\left.\begin{aligned}\frac{y-4}{4}&=\frac{x+1}{10}\\ \frac{x}{6}+\frac{y+2}{5}&=3\tfrac{1}{2}\end{aligned}\right\}$$

17. $$\left.\begin{aligned}\frac{5x-6y}{11}+2x&=3(y-1)\\ \frac{5x+6y}{10}-\frac{4x-3y}{3}&=y-2\end{aligned}\right\}$$

18. $$\left.\begin{aligned}\frac{4x-3y-7}{5}&=\frac{9x-4y-25}{30}\\ \frac{y-1}{3}+\frac{10x-3y-20}{20}&=\frac{3x+2y+3}{30}\end{aligned}\right\}$$

NOTE. In solving the following problems proceed as in § 175.

19. $$\left.\begin{aligned}\frac{6y+5}{8}-\frac{4x-5y+3}{4x-2y}&=\frac{9y-4}{12}\\ \frac{8x+3}{4}+\frac{x-3y}{7-x}&=\frac{6x-1}{3}\end{aligned}\right\}$$

20. $$\left.\begin{aligned}x-\frac{2y-x}{23-x}&=x-9\tfrac{1}{2}\\ y+\frac{y-3}{x-18}&=y-5\tfrac{2}{3}\end{aligned}\right\}$$

21. $$\left.\begin{aligned}\frac{4x+7}{3}+\frac{5x-4y}{2x+8}&=\frac{17+8x}{6}\\ \frac{5x-12}{4}-\frac{4x-6y-13}{2x-3y}&=\frac{10x-53}{8}\end{aligned}\right\}$$

22. $$\left.\begin{aligned}\frac{7+8x}{10}-\frac{3(x-2y)}{2(x-4)}&=\frac{11+4x}{5}\\ \frac{3(2y+3)}{4}&=\frac{6y+21}{4}-\frac{3y+5x}{2(2y-3)}\end{aligned}\right\}$$

194. Literal Simultaneous Equations.

Solve:
$$\left.\begin{aligned} ax + by &= c \\ a'x + b'y &= c' \end{aligned}\right\}$$

NOTE. The letters a', b' are read *a prime*, *b prime*. In like manner, a'', a''' are read *a second*, *a third*, and a_1, a_2, a_3 are read *a sub one*, *a sub two*, *a sub three*. It is sometimes convenient to represent different numbers that have a common property by the same letter marked by *accents* or *suffixes*. Here a and a' have a common property as coefficients of x.

$$ax + by = c. \qquad (1)$$
$$a'x + b'y = c'. \qquad (2)$$

To find the value of y, multiply (1) by a' and (2) by a.

$$aa'x + a'by = a'c$$
$$aa'x + ab'y = ac'$$

Subtract, $$a'by - ab'y = a'c - ac'$$

Divide by $a'b - ab'$, $$y = \frac{a'c - ac'}{a'b - ab'}.$$

To find the value of x, multiply (1) by b', and (2) by b.

$$ab'x + bb'y = b'c$$
$$a'bx + bb'y = bc'$$

Subtract, $$ab'x - a'bx = b'c - bc'$$

Divide by $ab' - a'b$, $$x = \frac{b'c - bc'}{ab' - a'b}.$$

EXERCISE 74.

Solve:

1. $\left.\begin{aligned} x + y &= s \\ x - y &= d \end{aligned}\right\}$

2. $\left.\begin{aligned} mx + ny &= r \\ m'x + n'y &= r' \end{aligned}\right\}$

3. $\left.\begin{aligned} ax - by &= c \\ a'x + b'y &= c' \end{aligned}\right\}$

4. $\left.\begin{aligned} x - y &= mn \\ cx + aby &= ms \end{aligned}\right\}$

5. $\left.\begin{aligned} bx + ay &= abc \\ x &= dy \end{aligned}\right\}$

6. $\left.\begin{aligned} bx + ay &= 1 \\ b'x - a'y &= 1 \end{aligned}\right\}$

7. $\left.\begin{aligned} 3\,bx + 2\,ay &= 3\,ab \\ 4\,bx - 3\,ay &= \tfrac{7}{6}\,ab \end{aligned}\right\}$

8. $\left.\begin{aligned} 2\,x - 3\,y &= a - b \\ 3\,x - 2\,y &= a + b \end{aligned}\right\}$

9. $$\left.\begin{array}{l}\dfrac{bx}{a^2-b^2}+\dfrac{cy}{b^2-a^2}=\dfrac{1}{a+b}\\ bx+cy=a+b\end{array}\right\}$$

10. $$\left.\begin{array}{l}\dfrac{x+m}{y-n}=\dfrac{a}{b}\\ bx+ay=c\end{array}\right\}$$

11. $$\left.\begin{array}{l}\dfrac{x}{a}+\dfrac{2y}{b}=1\\ \dfrac{2x}{a}-\dfrac{y}{b}=\dfrac{1}{3}\end{array}\right\}$$

12. $$\left.\begin{array}{l}\dfrac{x}{a+b}+\dfrac{y}{a-b}=2\\ x+y=2a\end{array}\right\}$$

13. $$\left.\begin{array}{l}\dfrac{2x}{3a}+\dfrac{4y}{3b}=\dfrac{3y}{b}-\dfrac{x}{a}\\ x-y=a-b\end{array}\right\}$$

14. $$\left.\begin{array}{l}ax+by=c\\ bx+ay=c\end{array}\right\}$$

15. $$\left.\begin{array}{l}3a^2+\ \ ax=b^2+by\\ \ \ ax+2by=d\end{array}\right\}$$

16. $$\left.\begin{array}{l}\dfrac{x}{a+b}-\dfrac{y}{a-b}=\dfrac{1}{a+b}\\ \dfrac{x}{a+b}+\dfrac{y}{a-b}=\dfrac{1}{a-b}\end{array}\right\}$$

17. $$\left.\begin{array}{l}\dfrac{x}{m-a}+\dfrac{y}{m-b}=1\\ \dfrac{x}{n-a}+\dfrac{y}{n-b}=1\end{array}\right\}$$

195. Fractional simultaneous equations, of which the denominators are simple expressions and contain the unknown numbers, may be solved as follows:

1. Solve:
$$\left.\begin{array}{ll}\dfrac{a}{x}+\dfrac{b}{y}=m & (1)\\ \dfrac{c}{x}+\dfrac{d}{y}=n & (2)\end{array}\right\}$$

To find the value of y.

Multiply (1) by c, $$\frac{ac}{x}+\frac{bc}{y}=cm. \qquad (3)$$

Multiply (2) by a, $$\frac{ac}{x}+\frac{ad}{y}=an. \qquad (4)$$

Subtract (4) from (3), $$\frac{bc-ad}{y}=cm-an.$$

Multiply both sides by y, and we have

$$bc - ad = (cm - an)y.$$

Divide by $cm - an$, $$y = \frac{bc - ad}{cm - an}.$$

To find the value of x.

Multiply (1) by d, $$\frac{ad}{x} + \frac{bd}{y} = dm. \qquad (5)$$

Multiply (2) by b, $$\frac{bc}{x} + \frac{bd}{y} = bn. \qquad (6)$$

Subtract (6) from (5), $$\frac{ad - bc}{x} = dm - bn.$$

Multiply both sides by x, and we have

$$ad - bc = (dm - bn)x.$$

Divide by $dm - bn$, $$x = \frac{ad - bc}{dm - bn}$$

2. Solve: $$\left.\begin{aligned} \frac{5}{3x} + \frac{2}{5y} &= 7 \qquad (1) \\ \frac{7}{6x} - \frac{1}{10y} &= 3 \qquad (2) \end{aligned}\right\}$$

Multiply (1) by 15, the L. C. M. of 3 and 5; and (2) by 30, the L. C. M. of 6 and 10,

$$\frac{25}{x} + \frac{6}{y} = 105. \qquad (3)$$

$$\frac{35}{x} - \frac{3}{y} = 90. \qquad (4)$$

Multiply (4) by 2, and add the result to (3),

$$\frac{95}{x} = 285.$$

$$\therefore x = \frac{1}{3}.$$

Substitute the value of x in (1), and we obtain

$$y = \frac{1}{5}.$$

Exercise 75.

Solve:

1. $$\left.\begin{aligned}\frac{5}{x}+\frac{6}{y}&=2\\ \frac{15}{x}-\frac{3}{y}&=2\tfrac{1}{2}\end{aligned}\right\}$$

2. $$\left.\begin{aligned}\frac{5}{x}+\frac{13}{y}&=49\\ \frac{7}{x}+\frac{3}{y}&=23\end{aligned}\right\}$$

3. $$\left.\begin{aligned}\frac{3}{x}+\frac{8}{y}&=3\\ \frac{15}{x}-\frac{4}{y}&=4\end{aligned}\right\}$$

4. $$\left.\begin{aligned}\frac{2}{x}+\frac{5}{y}&=19\\ \frac{8}{x}-\frac{3}{y}&=7\end{aligned}\right\}$$

5. $$\left.\begin{aligned}\frac{4}{5x}+\frac{5}{6y}&=5\tfrac{11}{15}\\ \frac{5}{4x}-\frac{4}{5y}&=\frac{11}{20}\end{aligned}\right\}$$

6. $$\left.\begin{aligned}\frac{1}{2x}+\frac{2}{3y}&=3\\ \frac{3}{4x}+\frac{4}{5y}&=3.9\end{aligned}\right\}$$

7. $$\left.\begin{aligned}\frac{1}{x}+\frac{1}{y}&=m\\ \frac{1}{x}-\frac{1}{y}&=n\end{aligned}\right\}$$

8. $$\left.\begin{aligned}\frac{m}{x}+\frac{1}{y}&=b\\ \frac{n}{x}+\frac{1}{y}&=c\end{aligned}\right\}$$

9. $$\left.\begin{aligned}\frac{m}{x}+\frac{n}{y}&=a\\ \frac{r}{x}+\frac{s}{y}&=b\end{aligned}\right\}$$

10. $$\left.\begin{aligned}\frac{a}{x}-\frac{b}{y}&=\frac{ac}{b}\\ \frac{b}{x}-\frac{a}{y}&=\frac{bc}{a}\end{aligned}\right\}$$

11. $$\left.\begin{aligned}\frac{3}{ax}-\frac{2}{by}&=5\\ \frac{2}{ax}-\frac{3}{by}&=2\end{aligned}\right\}$$

12. $$\left.\begin{aligned}\frac{a}{bx}+\frac{b}{ay}&=a+b\\ \frac{b}{x}+\frac{a}{y}&=a^2+b^2\end{aligned}\right\}$$

196. If three simultaneous equations are given, involving three unknown numbers, one of the unknowns must be eliminated between *two pairs* of the equations; then a second unknown between the two resulting equations.

Likewise, if four or more equations are given, involving four or more unknown numbers, one of the unknowns must be eliminated between three or more pairs of the equations; then a second between the pairs that can be formed of the resulting equations; and so on.

NOTE. The pairs chosen to eliminate from must be independent pairs, so that *each of the given equations* shall be used in the process of the eliminations.

Solve:

$$\left.\begin{aligned} 2x - 3y + 4z &= 4 \\ 3x + 5y - 7z &= 12 \\ 5x - y - 8z &= 5 \end{aligned}\right\} \qquad \begin{aligned}&(1)\\&(2)\\&(3)\end{aligned}$$

Eliminate z between the equations (1) and (3).

$$\begin{array}{lrl} \text{Multiply (1) by 2,} & 4x - 6y + 8z = 8 & (4) \\ \text{(3) is} & \underline{5x - y - 8z = 5} & \\ \text{Add,} & 9x - 7y = 13 & (5) \end{array}$$

Eliminate z between the equations (1) and (2).

$$\begin{array}{lrl} \text{Multiply (1) by 7,} & 14x - 21y + 28z = 28 & \\ \text{Multiply (2) by 4,} & \underline{12x + 20y - 28z = 48} & \\ \text{Add,} & 26x - y = 76 & (6) \end{array}$$

We now have two equations (5) and (6) involving two unknowns, x and y.

$$\begin{array}{lrl} \text{Multiply (6) by 7,} & 182x - 7y = 532 & (7) \\ \text{(5) is} & \underline{9x - 7y = 13} & \\ \text{Subtract (5) from (7),} & 173x = 519 & \\ & \therefore x = 3. & \end{array}$$

Substitute the value of x in (6), $78 - y = 76.$

$$\therefore y = 2.$$

Substitute the values of x and y in (1),

$$6 - 6 + 4z = 4.$$

$$\therefore z = 1.$$

EXERCISE 76.

Solve:

1. $$\left.\begin{aligned} x + y - 8 &= 0\\ y + z - 28 &= 0\\ x + z - 14 &= 0 \end{aligned}\right\}$$

2. $$\left.\begin{aligned} 4x + 3y + 2z &= 25\\ 3x - 2y + 5z &= 20\\ 10x - 5y + 3z &= 17 \end{aligned}\right\}$$

3. $$\left.\begin{aligned} 5x - 2y - 2z &= 12\\ x + y + z &= 8\\ 7x + 3y + 4z &= 42 \end{aligned}\right\}$$

4. $$\left.\begin{aligned} x - y + z &= 11\\ 3x + 3y - 2z &= 60\\ 10x - 5y - 3z &= 0 \end{aligned}\right\}$$

5. $$\left.\begin{aligned} 10x - y + 3z &= 42\\ 7x + 2y + z &= 51\\ 3x + 3y - z &= 24 \end{aligned}\right\}$$

6. $$\left.\begin{aligned} 5x + 2y - 3z &= 160\\ 3x + 9y + 8z &= 115\\ 2x - 3y - 5z &= 45 \end{aligned}\right\}$$

7. $$\left.\begin{aligned} 6x - 2y + 5z &= 53\\ 5x + 3y + 7z &= 33\\ x + y + z &= 5 \end{aligned}\right\}$$

8. $$\left.\begin{aligned} 3x - 3y + 4z &= 20\\ 6x + 2y - 7z &= 5\\ 2x - y + 8z &= 45 \end{aligned}\right\}$$

9. $$\left.\begin{aligned} 2x + 7y + 10z &= 25\\ x + y - z &= 9\\ 7x - 7y - 11z &= 73 \end{aligned}\right\}$$

10. $$\left.\begin{aligned} 5x + 2y - 20z &= 20\\ 3x - 6y + 7z &= 51\\ 4x + 8y - 9z &= 53 \end{aligned}\right\}$$

11. $$\left.\begin{aligned} x + 2y + 10z &= 44\\ 3x + 3y + 7z &= 384\\ 2x + y + z &= 256 \end{aligned}\right\}$$

12. $$\left.\begin{aligned} 10x &= y + 4z + 56\\ 3y &= 2x + 3z - 98\\ 2z &= x - 3y - 18 \end{aligned}\right\}$$

13. $$\left.\begin{aligned} 3x - 5y - 2z &= 14\\ 5x - 8y - z &= 12\\ x - 3y - 3z &= 1 \end{aligned}\right\}$$

14. $$\left.\begin{aligned} 2x + 3y + z &= 31\\ x - y + 3z &= 13\\ 10y + 5x - 2z &= 48 \end{aligned}\right\}$$

15. $$\left.\begin{aligned} 2x + 3y - 4z &= 1\\ 10x - 6y + 12z &= 6\\ x + 12y + 2z &= 5 \end{aligned}\right\}$$

16. $$\left.\begin{aligned} 3x + 6y + 2z &= 3\\ 12y + 4z - 6x &= 2\\ 9x + 18y - 4z &= 4 \end{aligned}\right\}$$

17. $$\left.\begin{aligned} 2x + y + 2z &= 3\\ 5y - 4x - 4z &= 1\\ 3x + 9y + z &= 9 \end{aligned}\right\}$$

18. $$\left.\begin{aligned} 3x + 2y + z &= 20\tfrac{1}{2}\\ 2x - y + 3z &= 26\tfrac{1}{2}\\ x + y + 10z &= 55 \end{aligned}\right\}$$

19. $$\left.\begin{aligned} \frac{1}{x}-\frac{2}{y}+4&=0 \\ \frac{1}{y}-\frac{1}{z}+1&=0 \\ \frac{2}{z}+\frac{3}{x}-14&=0 \end{aligned}\right\}$$

20. $$\left.\begin{aligned} \frac{1}{x}+\frac{1}{y}+\frac{1}{z}&=36 \\ \frac{1}{x}+\frac{3}{y}-\frac{1}{z}&=28 \\ \frac{1}{x}+\frac{1}{3y}+\frac{1}{2z}&=20 \end{aligned}\right\}$$

21. $$\left.\begin{aligned} \frac{1}{x}+\frac{2}{y}-\frac{3}{z}&=1 \\ \frac{5}{x}+\frac{4}{y}+\frac{6}{z}&=24 \\ \frac{7}{x}-\frac{8}{y}+\frac{9}{z}&=14 \end{aligned}\right\}$$

22. $$\left.\begin{aligned} \frac{1}{x}+\frac{1}{y}-\frac{1}{z}&=a \\ \frac{1}{x}-\frac{1}{y}+\frac{1}{z}&=b \\ \frac{1}{y}+\frac{1}{z}-\frac{1}{x}&=c \end{aligned}\right\}$$

23. $$\left.\begin{aligned} \frac{4}{x}-\frac{3}{y}&=\frac{1}{20} \\ \frac{2}{z}-\frac{3}{x}&=\frac{1}{15} \\ \frac{4}{z}-\frac{5}{y}&=\frac{1}{12} \end{aligned}\right\}$$

24. $$\left.\begin{aligned} \frac{1}{x}+\frac{1}{y}-\frac{1}{a}&=0 \\ \frac{1}{x}+\frac{1}{z}-\frac{1}{b}&=0 \\ \frac{1}{y}+\frac{1}{z}-\frac{1}{c}&=0 \end{aligned}\right\}$$

25. $$\left.\begin{aligned} \frac{x}{2}+\frac{y}{3}+\frac{z}{4}&=62 \\ \frac{x}{3}+\frac{y}{4}+\frac{z}{5}&=47 \\ \frac{x}{4}+\frac{y}{5}+\frac{z}{6}&=38 \end{aligned}\right\}$$

26. $$\left.\begin{aligned} \frac{2}{x}+\frac{1}{y}-\frac{3}{z}&=0 \\ \frac{3}{z}-\frac{2}{y}-2&=0 \\ \frac{1}{x}+\frac{1}{z}-\frac{4}{3}&=0 \end{aligned}\right\}$$

27. $$\left.\begin{aligned} \frac{15}{x}-\frac{4}{y}+\frac{5}{z}&=38 \\ \frac{2}{x}+\frac{3}{y}+\frac{12}{z}&=61 \\ \frac{8}{x}-\frac{5}{y}+\frac{40}{z}&=161 \end{aligned}\right\}$$

28. $$\left.\begin{aligned} \frac{2}{x}-\frac{3}{y}+\frac{4}{z}&=2.9 \\ \frac{5}{x}-\frac{6}{y}-\frac{7}{z}&=-10.4 \\ \frac{9}{y}+\frac{10}{z}-\frac{8}{x}&=14.9 \end{aligned}\right\}$$

CHAPTER XII.

PROBLEMS INVOLVING TWO OR MORE UNKNOWN NUMBERS.

197. It is often necessary in the solution of problems to employ two or more letters to represent the numbers to be found. In all cases the conditions must be sufficient to give just as many equations as there are unknown numbers employed. If there are *more* equations than unknown numbers, some of them are superfluous or inconsistent; if there are *fewer* equations than unknown numbers, the problem is indeterminate.

EXERCISE 77.

1. If A gave B \$10, B would have three times as much money as A. If B gave A \$10, A would have twice as much money as B. How much has each?

Let $x =$ the number of dollars A has,
and $y =$ the number of dollars B has.

Then, if A gave B \$10,

$x - 10 =$ the number of dollars A would have,
$y + 10 =$ the number of dollars B would have.

$$\therefore y + 10 = 3(x - 10). \qquad (1)$$

If B gave A \$10,

$x + 10 =$ the number of dollars A would have,
$y - 10 =$ the number of dollars B would have.

$$\therefore x + 10 = 2(y - 10). \qquad (2)$$

From the solution of equations (1) and (2), $x = 22$, and $y = 26$.

Therefore, A has \$22, and B has \$26.

2. If the smaller of two numbers is divided by the greater, the quotient is 0.21, and the remainder 0.0057; but if the greater is divided by the smaller, the quotient is 4 and the remainder 0.742. Find the numbers.

Let $x =$ the greater number,

and $y =$ the smaller number.

Then, $$\frac{y - 0.0057}{x} = 0.21, \tag{1}$$

and $$\frac{x - 0.742}{y} = 4. \tag{2}$$

$$\therefore y - 0.21x = 0.0057, \tag{3}$$

$$x - 4y = 0.742. \tag{4}$$

Multiply (3) by 4, $$4y - 0.84x = 0.0228 \tag{5}$$

(4) is $$-4y + x = 0.742$$

Add, $$0.16x = 0.7648$$

$$\therefore x = 4.78.$$

Put the value of x in (2), $$4y = 4.038,$$

$$\therefore y = 1.0095.$$

Therefore, the numbers are 4.78 and 1.0095.

3. If A gave B $100, A would then have half as much money as B; but if B gave A $100, B would have one third as much as A. How much has each?

4. If the greater of two numbers is divided by the smaller, the quotient is 7 and the remainder 4; but if three times the greater number is divided by twice the smaller, the quotient is 11 and the remainder 4. Find the numbers.

5. If the greater of two numbers is divided by the smaller, the quotient is 4 and the remainder 0.37; but if the smaller is divided by the greater, the quotient is 0.23 and the remainder 0.0149. Find the numbers.

6. If A gave B $5, he would have $6 less than B; but if he received $5 from B, three times his money would be $20 more than four times B's. How much has each?

7. If the numerator of a fraction is doubled and its denominator diminished by 1, its value will be $\frac{1}{2}$. If its denominator is doubled and its numerator increased by 1, its value will be $\frac{1}{7}$. Find the fraction.

Let x = the numerator,
and y = the denominator.

Then,
$$\frac{2x}{y-1}=\frac{1}{2}, \qquad (1)$$
and
$$\frac{x+1}{2y}=\frac{1}{7}. \qquad (2)$$

The solution of equations (1) and (2) gives 5 for x and 21 for y.

Therefore, the required fraction is $\frac{5}{21}$.

8. A certain fraction becomes equal to $\frac{1}{2}$ if 3 is added to its numerator and 1 to its denominator, and equal to $\frac{1}{4}$ if 3 is subtracted from its numerator and from its denominator. Find the fraction.

9. A certain fraction becomes equal to $\frac{9}{11}$ if 1 is added to double its numerator, and equal to $\frac{1}{3}$ if 3 is subtracted from its numerator and from its denominator. Find the fraction.

10. There are two fractions with numerators 11 and 5, respectively, whose sum is $1\frac{4}{9}$; but if their denominators are interchanged their sum is $2\frac{1}{9}$. Find the fractions.

11. A certain fraction becomes equal to $\frac{1}{2}$ when 7 is added to its denominator, and equal to 2 when 13 is added to its numerator. Find the fraction.

12. A certain fraction becomes equal to $\frac{7}{9}$ when the denominator is increased by 4, and equal to $\frac{20}{41}$ when the numerator is diminished by 15. Find the fraction.

13. A certain fraction becomes equal to $\frac{2}{3}$ if 7 is added to the numerator, and equal to $\frac{3}{8}$ if 7 is subtracted from the denominator. Find the fraction.

14. A certain number is expressed by three digits. The sum of the digits is 21. The sum of the first and last digits is twice the middle digit. If the hundreds' and tens' digits are interchanged, the number is diminished by 90. Find the number.

Let $x =$ the hundreds' digit,
$y =$ the tens' digit,
$z =$ the units' digit.

Then, $100x + 10y + z =$ the number.

By the conditions,

$$x + y + z = 21, \tag{1}$$

$$x + z = 2y, \tag{2}$$

and

$$100y + 10x + z = 100x + 10y + z - 90. \tag{3}$$

Solving these equations, $x = 8$, $y = 7$, $z = 6$.

Therefore, the number is 876.

15. The sum of the two digits of a number is 9, and if 27 is subtracted from the number, the digits will be reversed. Find the number.

16. The sum of the two digits of a number is 9, and if the number is divided by the sum of the digits, the quotient is 5. Find the number.

17. A certain number is expressed by two digits. The sum of the digits is 11. If the digits are reversed, the new number exceeds the given number by 27. Find the number.

18. A certain number is expressed by three digits, the units' digit being zero. If the hundreds' and tens' digits are interchanged, the number is diminished by 180. If the hundreds' digit is halved, and the tens' and units' digits are interchanged, the number is diminished by 336. Find the number.

19. A number is expressed by three digits. If the digits are reversed, the new number exceeds the given number by 99. If the number is divided by nine times the sum of its digits, the quotient is 3. The sum of the hundreds' and units' digits exceeds the tens' digit by 1. Find the number.

20. A boatman rows 20 miles down a river and back in 8 hours. He finds that he can row 5 miles down the river in the same time that he rows 3 miles up the river. Find the time he was rowing down and up respectively.

Let $x =$ the boatman's rate per hour in still water,
and $y =$ the rate per hour of the current.

Then, $\frac{20}{x+y} =$ the number of hours he was rowing down,
and $\frac{20}{x-y} =$ the number of hours he was rowing up.

Therefore,
$$\frac{20}{x+y} + \frac{20}{x-y} = 8, \tag{1}$$
and
$$\frac{5}{x+y} = \frac{3}{x-y}. \tag{2}$$

Solving these equations, $x = 5\frac{1}{3}$, $y = 1\frac{1}{3}$.

Therefore,
$$\frac{20}{x+y} = 3,\ \frac{20}{x-y} = 5.$$

It takes him 3 hours to row down and 5 hours to row back.

21. A boat's crew which can pull down a river at the rate of 10 miles an hour finds that it takes twice as long to row a mile up the river as to row a mile down. Find the rate of their rowing in still water and the rate of the stream.

22. A boatman rows down a stream, which runs at the rate of $2\frac{1}{2}$ miles an hour, for a certain distance in 1 hour 30 minutes; it takes him 4 hours 30 minutes to return. Find the distance he pulled down the stream and his rate of rowing in still water.

23. A person rows down a stream a distance of 20 miles and back again in 10 hours. He finds he can row 2 miles against the stream in the same time he can row 3 miles with it. Find the time of his rowing down and of his rowing up the stream; and also the rate of the stream.

24. A and B can do a piece of work together in 3 days, A and C in 4 days, B and C in $4\frac{1}{2}$ days. How long will it take each alone to do the work?

Let x, y, $z =$ the number of days in which A, B, C can do the work, respectively.

Then, $\frac{1}{x}, \frac{1}{y}, \frac{1}{z} =$ the parts A, B, C can do in 1 day, respectively.

And $\frac{1}{x} + \frac{1}{y} =$ the part A and B together can do in one day.

But $\frac{1}{3} =$ the part A and B together can do in 1 day.

Therefore,
$$\frac{1}{x} + \frac{1}{y} = \frac{1}{3}, \tag{1}$$

Likewise,
$$\frac{1}{x} + \frac{1}{z} = \frac{1}{4}, \tag{2}$$

and
$$\frac{1}{y} + \frac{1}{z} = \frac{2}{9}, \quad \left(\frac{1}{4\frac{1}{2}} = \frac{2}{9}\right) \tag{3}$$

Add, and divide by 2,
$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{29}{72} \tag{4}$$

Subtract (1), (2), and (3), separately from (4), and we have

$$\frac{1}{z} = \frac{5}{72}, \ \frac{1}{y} = \frac{11}{72}, \ \frac{1}{x} = \frac{13}{72}.$$

Therefore, $z = 14\frac{2}{5}$, $y = 6\frac{6}{11}$, $x = 5\frac{7}{13}$.

Therefore, A can do the work in $5\frac{7}{13}$ days, B in $6\frac{6}{11}$ days, and C in $14\frac{2}{5}$ days.

25. A cistern has three pipes, A, B, and C. A and B will fill the cistern in 1 hour 10 minutes, A and C in 1 hour 24 minutes, B and C in 2 hours 20 minutes. How long will it take each pipe alone to fill it?

26. A and B can do a piece of work in $2\frac{1}{2}$ days, A and C in $3\frac{1}{3}$ days, B and C in 4 days. How long will it take each alone to do the work?

27. A and B can do a piece of work in a days, A and C in b days, B and C in c days. How long will it take each alone to do the work?

28. A sum of money, at simple interest, amounted to \$2480 in 4 years, and to \$2600 in 5 years. Find the sum and the rate of interest.

Let $x =$ the number of dollars in the principal,
and $y =$ the rate of interest.

The interest for one year is $\frac{y}{100}$ of the principal, $= \frac{y}{100}$ of x; for 4 years, $= \frac{4y}{100}$ of x; and for 5 years, $= \frac{5y}{100}$ of x.

The amount is principal + interest.

Therefore,
$$x + \frac{4xy}{100} = 2480.$$
$$x + \frac{5xy}{100} = 2600.$$

Hence,
$$100x + 4xy = 248,000. \quad (1)$$
$$100x + 5xy = 260,000. \quad (2)$$

Multiply (1) by 5 and (2) by 4, and we have
$$500x + 20xy = 1,240,000$$
$$400x + 20xy = 1,040,000$$

Subtract,
$$100x = 200,000.$$

Therefore,
$$x = 2000.$$

Substitute the value of x in (1), $y = 6$.

Therefore, the sum is \$2000, and the rate 6%.

29. A sum of money, at simple interest, amounted in 4 years to \$29,000, and in 5 years to \$30,000. Find the sum and the rate of interest.

30. A sum of money, at simple interest, amounted in 10 months to \$2100, and in 18 months to \$2180. Find the sum and the rate of interest.

31. A man has \$10,000 invested. For a part of this sum he receives 5 per cent interest, and for the rest 6 per cent; the income from his 5 per cent investment is \$60 more than from his 6 per cent. How much has he in each investment?

32. In a mile race A gives B a start of 20 yards and beats him by 30 seconds. At the second trial A gives B a start of 32 seconds and beats him by $9\frac{5}{11}$ yards. Find the number of yards each runs a second.

Let $x =$ the number of yards A runs a second,

and $y =$ the number of yards B runs a second.

Since there are 1760 yards in a mile,

$$\frac{1760}{x} = \text{the number of seconds it takes A to run a mile.}$$

Since B has a start of 20 yards, he runs 1740 yards the first trial; and as he was 30 seconds longer than A,

$$\frac{1760}{x} + 30 = \text{the number of seconds B was running.}$$

But $\frac{1740}{y} = \text{the number of seconds B was running.}$

$$\therefore \frac{1740}{y} = \frac{1760}{x} + 30. \qquad (1)$$

In the second trial B runs $(1760 - 9\frac{5}{11})$ yards $= 1750\frac{6}{11}$ yards.

$$\therefore \frac{1750\frac{6}{11}}{y} = \frac{1760}{x} + 32. \qquad (2)$$

From the solution of equations (1) and (2), $x = 5\frac{13}{15}$, and $y = 5\frac{3}{11}$.

Therefore, A runs $5\frac{13}{15}$ yards a second, and B runs $5\frac{3}{11}$ yards a second.

33. Two men, A and B, run a mile, and A wins by 2 seconds. In the second trial B has a start of $18\frac{1}{3}$ yards, and wins by 1 second. Find the number of yards each runs a second, and the number of miles each would run in an hour.

34. In a mile race A gives B a start of 3 seconds, and is beaten by $12\frac{4}{7}$ yards. In the second trial A gives B a start of 10 yards, and the race is a tie. Find the number of yards each runs a second. At this rate, how many miles could each run in an hour?

35. A train, after traveling an hour from A towards B, meets with an accident which detains it half an hour; after which it proceeds at four fifths of its usual rate, and arrives an hour and a quarter late. If the accident had happened 30 miles farther on, the train would have been only an hour late. Find the usual rate of the train.

Let $y =$ the number of miles from A to B,
and $5x =$ the number of miles the train travels per hour.
Then, $4x =$ the rate of the train after the accident.
Then, $y - 5x =$ the number of miles the train has to go after the accident.

Hence, $\frac{y-5x}{5x} =$ the number of hours required usually,
and $\frac{y-5x}{4x} =$ the number of hours actually required.

$$\therefore \frac{y-5x}{4x} - \frac{y-5x}{5x} = \text{the loss in hours of running time.}$$

But since the train was detained $\frac{1}{2}$ an hour and arrived $1\frac{1}{4}$ hours late, the *running time* was $\frac{3}{4}$ of an hour more than usual. That is,

$$\tfrac{3}{4} = \text{loss in hours of running time.}$$

$$\therefore \frac{y-5x}{4x} - \frac{y-5x}{5x} = \frac{3}{4}. \quad (1)$$

If the accident had happened 30 miles farther on, the remainder of the journey would have been $y - (5x + 30)$ miles, and the loss in running time would have been $\frac{1}{2}$ an hour.

$$\therefore \frac{y-(5x+30)}{4x} - \frac{y-(5x+30)}{5x} = \frac{1}{2}. \quad (2)$$

From the solution of equations (1) and (2), $x = 6$, and $5x = 30$.

Therefore, the usual rate of the train is 30 miles an hour.

36. An express train, after traveling an hour from A towards B, meets with an accident which delays it 15 minutes. It afterwards proceeds at two thirds its usual rate, and arrives 24 minutes late. If the accident had happened 5 miles farther on, the train would have been only 21 minutes late. Find the usual rate of the train.

37. If 3 yards of velvet and 12 yards of silk cost $60, and 4 yards of velvet and 5 yards of silk cost $58, what is the price of a yard of velvet and of a yard of silk?

38. If 5 bushels of wheat, 4 of rye, and 3 of oats are sold for $9; 3 bushels of wheat, 5 of rye, and 6 of oats for $8.75; and 2 bushels of wheat, 3 of rye, and 9 of oats for $7.25; what is the price per bushel of each kind of grain?

39. A train proceeded a certain distance at a uniform rate. If the speed had been 6 miles an hour more, the time occupied would have been 5 hours less; but if the speed had been 6 miles an hour less, the time occupied would have been $7\frac{1}{2}$ hours more. Find the distance.

HINT. If $x =$ the number of hours the train travels, and y the number of miles per hour, then $xy =$ the distance.

40. A certain number of persons paid a bill. If there had been 10 persons more, each would have paid $2 less; but if there had been 5 persons less, each would have paid $2.50 more. Find the number of persons and the amount of the bill.

41. A man bought 10 cows and 50 sheep for $750. He sold the cows at a profit of 10 per cent, and the sheep at a profit of 30 per cent, and received in all $875. Find the average cost of a cow and of a sheep.

42. It is 40 miles from Dover to Portland. A sets out from Dover, and B from Portland, at 7 o'clock A.M., to meet each other. A walks at the rate of $3\frac{1}{2}$ miles an hour, but stops 1 hour on the way; B walks at the rate of $2\frac{1}{2}$ miles an hour. At what time of day and how far from Portland will they meet?

43. A number is expressed by three digits. The sum of the digits is 21; the sum of the first and second exceeds the third by 3; and if 198 is added to the number, the digits in the units' and hundreds' places will be interchanged. Find the number.

44. If the length of a rectangular field is increased by 5 yards and its breadth by 10 yards, its area is increased by 450 square yards; but if its length is increased by 5 yards and its breadth diminished by 10 yards, its area is diminished by 350 square yards. Find its dimensions.

45. If the floor of a certain hall had been 2 feet longer and 4 feet wider, it would have contained 528 square feet more; but if the length and width were each 2 feet less, it would contain 316 square feet less. Find its dimensions.

46. If the length of a rectangle was 4 feet less and the width 3 feet more, the figure would be a square of the same area as the given rectangle. Find the dimensions of the rectangle.

47. If a certain number is divided by the sum of its two digits diminished by 2, the quotient is 5 and the remainder 1; if the digits are interchanged, and the resulting number is divided by the sum of the digits increased by 2, the quotient is 5 and the remainder 8. Find the number.

48. A person has a certain capital invested at a certain rate per cent. Another person has $2000 more capital invested at one per cent better than the first, and receives $150 more income. A third person has $3000 more capital invested at two per cent better than the first, and receives $280 more income. Find the capital of each, and the rate at which it is invested.

49. A man makes an investment at 4 per cent, and a second investment at $4\frac{1}{2}$ per cent. His income from the two investments is $715. If the first investment had been made at $4\frac{1}{2}$ per cent and the second at 4 per cent, his income would have been $15 greater. Find the amount of each investment.

50. A number is expressed by two digits, the units' digit being the larger. If the number is divided by the sum of its digits, the quotient is 4. If the digits are reversed and the resulting number is divided by 2 more than the difference of the digits, the quotient is 14. Find the number.

51. An income of \$335 a year is obtained from two investments, one in $4\frac{1}{2}$ per cent stock and the other in 5 per cent stock. If the $4\frac{1}{2}$ per cent stock should be sold at 110, and the 5 per cent at 125, the sum realized from both stocks together would be \$8300. How much of each stock is there?

52. A sum of money, at simple interest, amounted in m years to c dollars, and in n years to d dollars. Find the sum and the rate of interest.

53. A sum of money, at simple interest, amounted in m months to a dollars, and in n months to b dollars. Find the sum and the rate of interest.

54. A person has \$18,375 to invest. He can buy 3 per cent bonds at 75, and 5 per cent bonds at 120. How much of his money must he invest in each kind of bonds in order to have the same income from each investment?

HINT. Notice that the 3 per cent bonds at 75 pay 4 per cent on the money invested, and 5 per cent bonds at 120 pay $4\frac{1}{6}$ per cent.

55. In a mile race A gives B a start of 44 yards, and is beaten by 1 second. In a second trial A gives B a start of 6 seconds, and beats him by $9\frac{7}{9}$ yards. Find the number of yards each runs a second.

56. A train, after running 2 hours from A towards B, meets with an accident which delays it 20 minutes. It afterwards proceeds at four fifths its usual rate, and arrives 1 hour 40 minutes late. If the accident had happened 40 miles nearer A, the train would have been 2 hours late. Find the usual rate of the train.

57. A boy bought some apples at 3 for 5 cents, and some at 4 for 5 cents, paying $1 for the whole. He sold them at 2 cents apiece, and cleared 40 cents. How many of each kind did he buy?

58. Find the area of a rectangular floor, such that if 3 feet were taken from the length and 3 feet added to the breadth, its area would be increased by 6 square feet, but if 5 feet were taken from the breadth and 3 feet added to the length, its area would be diminished by 90 square feet.

59. A courier was sent from A to B, a distance of 147 miles. After 7 hours, a second courier was sent from A, who overtook the first just as he was entering B. The time required by the first to travel 17 miles added to the time required by the second to travel 76 miles is 9 hours 40 minutes. How many miles did each travel per hour?

60. A box contains a mixture of 6 quarts of oats and 9 of corn, and another box contains a mixture of 6 quarts of oats and 2 of corn. How many quarts must be taken from each box in order to have a mixture of 7 quarts, half oats and half corn?

61. A train traveling 30 miles an hour takes 21 minutes longer to go from A to B than a train which travels 36 miles an hour. Find the distance from A to B.

62. A man buys 570 oranges, some at 16 for 25 cents, and the rest at 18 for 25 cents. He sells them all at the rate of 15 for 25 cents, and gains 75 cents. How many of each kind does he buy?

63. A and B run a mile race. In the first heat B receives 12 seconds start, and is beaten by 44 yards. In the second heat B receives 165 yards start, and arrives at the winning post 10 seconds before A. Find the time in which each can run a mile.

198. The *discussion* of a problem consists in making various suppositions as to the relative values of the given numbers, and explaining the results. We will illustrate by the following example:

Two couriers are traveling along the same road, in the same direction. A travels m miles an hour, and B travels n miles an hour. At 12 o'clock B is d miles in advance of A. When will the couriers be together?

Suppose they will be together x hours *after* 12. Then A has traveled mx miles, and B has traveled nx miles, and as A has traveled d miles more than B

$$mx - nx = d. \qquad \therefore x = \frac{d}{m-n}.$$

Discussion. 1. If m is greater than n, the value of x is positive, and A will overtake B *after* 12 o'clock.

2. If m is less than n, the value of x is negative. In this case B travels faster than A, and as he is d miles ahead of A at 12 o'clock, A cannot overtake B *after* 12 o'clock, but B passed A *before* 12 o'clock. The supposition, therefore, that the couriers are together *after* 12 o'clock is incorrect, and the *negative* value of x points to an **error in the supposition.**

3. If m equals n, then the value of x assumes the form $\frac{d}{0}\cdot$ Now, if the couriers are d miles apart at 12 o'clock, and if they travel at the same rates, it is obvious that they never will be together, so that the symbol $\frac{d}{0}$ may be regarded as the **symbol of impossibility.**

4. If m equals n and d is 0, then $\frac{d}{m-n}$ becomes $\frac{0}{0}\cdot$ Now, if the couriers are together at 12 o'clock, and if they travel at the same rates, it is obvious that they will be together all the time, so that x may have an *indefinite number of values.* Hence, the symbol $\frac{0}{0}$ may be regarded as the **symbol of indetermination.**

EXERCISE 78.

1. A train traveling b miles per hour is m hours in advance of a second train that travels a miles per hour. In how many hours will the second train overtake the first?

Ans. $\frac{bm}{a-b}$.

Discuss the problem (1) when a is greater than b; (2) when a is equal to b; (3) when a is less than b.

2. A man setting out on a journey drove at the rate of a miles an hour to the nearest railway station, distant b miles from his house. On arriving at the station he found that the train left c hours before. At what rate per hour should he have driven in order to reach the station just in time for the train?

Ans. $\frac{ab}{b-ac}$.

Discuss the problem (1) when $c=0$; (2) when $c=\frac{b}{a}$; (3) when $c=-\frac{b}{a}$. In case (2), how many hours did the man have to drive from his house to the station? In case (3), what is the meaning of the *negative* value of c?

3. A wine merchant has two kinds of wine which he sells, one at a dollars, and the other at b dollars per gallon. He wishes to make a mixture of l gallons, that shall cost him on the average m dollars a gallon. How many gallons must he take of each?

Ans. $\frac{(m-b)l}{a-b}$ of the first; $\frac{(a-m)l}{a-b}$ of the second.

Discuss the problem (1) when $a=b$; (2) when a or $b=m$; (3) when $a=b=m$; (4) when a is greater than b and less than m; (5) when a is greater than b and b is greater than m.

CHAPTER XIII.

SIMPLE INDETERMINATE EQUATIONS.

199. If a *single* equation is given with *two* unknown numbers, and no other condition is imposed, the number of its solutions is *unlimited;* for, if any value is assigned to one of the unknown numbers a *corresponding* value may be found for the other. An equation that has an indefinite number of solutions is said to be **indeterminate.**

200. The values of the unknown numbers in an indeterminate equation are *dependent upon each other;* so that they are confined to a *particular range.*

This range may be still further limited by requiring these values to satisfy some given condition; as, for instance, that they shall be *positive integers.*

1. Solve $3x + 4y = 22$, in positive integers.

Transpose, $3x = 22 - 4y.$

Divide by 3, $x = 7 - y + \frac{1-y}{3}.$

Transpose, $x + y - 7 = \frac{1-y}{3}.$

Since x and y are integers, $\frac{1-y}{3}$ is an integer.

Let $\frac{1-y}{3} = m$, an integer.

Then, $y = 1 - 3m.$ (1)

Put this value for y in the given equation.

Then, $x = 6 + 4m.$ (2)

In equations (1) and (2),

If $m = 0$, then $y = 1$ and $x = 6.$

If $m = -1$, then $y = 4$ and $x = 2.$

No other value of m gives *positive* integers for *both* x and y.

2. Solve $5x - 14y = 11$, in positive integers.

Transpose, $$5x = 11 + 14y.$$

Divide by 5, $$x = 2 + 2y + \frac{1 + 4y}{5}.$$

Transpose, $$x - 2y - 2 = \frac{1 + 4y}{5}.$$

Then, $\dfrac{1 + 4y}{5}$ must be integral.

Let $\dfrac{1 + 4y}{5} = m$, then $y = \dfrac{5m - 1}{4}$ a fraction in form.

To avoid this difficulty, it is necessary to make the coefficient of y equal to *unity*. Since $\dfrac{1 + 4y}{5}$ is integral, any multiple of $\dfrac{1 + 4y}{5}$ is integral. Multiply the numerator of the fraction, then, by a number that will make the division of the coefficient of y *give a remainder of 1*. In this case, multiply by 4.

We have $$\frac{4 + 16y}{5} = 3y + \frac{4 + y}{5}.$$

Let $$\frac{4 + y}{5} = m, \text{ an integer.}$$

Then, $$y = 5m - 4. \qquad (1)$$

Since $x = \frac{1}{5}(11 + 14y)$, from the original equation,
$$x = 14m - 9. \qquad (2)$$

Here it is obvious that m may have *any positive value.*

If $m = 1$, $x = 5$, $y = 1$;
If $m = 2$, $x = 19$, $y = 6$;
If $m = 3$, $x = 33$, $y = 11$;
and so on.

3. Solve $5x + 6y = 30$, so that x may be a multiple of y, and both positive.

Let $$x = my.$$

Put this value of x in the given equation.

Then, $$(5m + 6)y = 30.$$

$$\therefore y = \frac{30}{5m + 6};$$

and $$x = \frac{30m}{5m + 6}.$$

If $m = 2$, $x = 3\frac{3}{4}$, $y = 1\frac{7}{8}$;
If $m = 3$, $x = 4\frac{2}{7}$, $y = 1\frac{3}{7}$.

EXERCISE 79.

Solve in positive integers:

1. $2x + 11y = 49.$
2. $7x + 3y = 40.$
3. $5x + 7y = 53.$
4. $x + 10y = 29.$
5. $3x + 8y = 61.$
6. $8x + 5y = 97.$
7. $16x + 7y = 110.$
8. $7x + 10y = 206.$

Solve in least positive integers:

9. $12x - 7y = 1.$
10. $5x - 17y = 23.$
11. $23y - 13x = 3.$
12. $23x - 9y = 929.$
13. $23x - 33y = 43.$
14. $555x - 22y = 73.$

15. A man spent $114 in buying calves at $5 apiece, and pigs at $3 apiece. How many did he buy of each?

16. In how many ways can a man pay a debt of $87 with five-dollar bills and two-dollar bills?

17. Find the smallest number that, when divided by 5 or when divided by 7, gives 4 for a remainder.

Let n = the number, then $\frac{n-4}{5} = x$, and $\frac{n-4}{7} = y$.

18. A farmer sold 15 calves, 14 lambs, and 13 pigs for $200. Some days after, at the same price for each kind, he sold 7 calves, 11 lambs, and 16 pigs, and received $141. What was the price of each?

First eliminate one of the unknowns from the two equations.

19. A number is expressed by three digits. The sum of the digits is 20. If 16 is subtracted from the number and the remainder divided by 2, the digits will be reversed. Find the number.

20. In how many ways may 100 be divided into two parts, one of which shall be a multiple of 7 and the other a multiple of 9?

CHAPTER XIV.

INEQUALITIES.

201. If $a - b$ is positive, a is said to be greater than b; if $a - b$ is negative, a is said to be less than b.

NOTE. Letters in this chapter are understood to stand for *positive* numbers, unless the contrary is expressly stated.

202. **An Inequality** is a statement in symbols that one of two numbers is greater than or less than the other.

203. **The Sign of an Inequality** is $>$, which always points toward the smaller number. Thus, $a > b$ is read a is greater than b; $c < d$ is read c is less than d.

204. The expressions that precede and follow the sign of an inequality are called, respectively, the **first and second members** of the inequality.

205. Two inequalities are said to **subsist in the same sense** if the signs of the inequalities point in the same direction; and two inequalities are said to be the **reverse of each other** if the signs point in opposite directions.

Thus, $a > b$ and $c > d$ subsist in the same sense, but $a > b$ and $c < d$ are the reverse of each other.

206. *If the signs of all the terms of an inequality are changed, the inequality is reversed.* Thus, if $a > b$, then $-a < -b$.

207. *If the members of an inequality are interchanged, the inequality is reversed.* Thus, if $a > b$, then $b < a$.

208. *An inequality will continue to subsist in the same sense if each member is increased, diminished, multiplied, or divided, by the* ***same positive number.***

Thus, if $a > b$, then $a + c > b + c$; $a - c > b - c$; $ac > bc$; $a \div c > b \div c$. Therefore,

209. ***A term can be transposed*** *from one member of an inequality to the other, provided the sign of the term is changed.*

Thus, if $a - c > b$,
by adding c to both members, $a > b + c$. (§ 208)

210. *An inequality will be reversed if its members are subtracted from equal numbers; or if its members are multiplied or divided by the* ***same negative number.***

Thus, if $x = y$ and $a > b$, then $x - a < y - b$; $-ac < -bc$; and $a \div (-c) < b \div (-c)$.

211. ***The sum or product*** *of the corresponding members of two inequalities that subsist in the same sense is an inequality in the same sense.*

Thus, if $a > b$ and $c > d$, then $a + c > b + d$, and $ac > bd$.

212. ***The difference or quotient*** *of the corresponding members of two inequalities that subsist in the same sense may be an inequality in the same sense, or the reverse, or may be an equality.*

$$\begin{array}{lc@{\qquad}lc} \text{Thus,} & 7 > 4 & & 5 > 4 \\ \text{By subtraction,} & \underline{3 > 2} & \text{By subtraction,} & \underline{3 > 2} \\ & 4 > 2 & & 2 = 2 \end{array}$$

$$\begin{array}{lc@{\qquad}lc} & 7 > 4 & & 5 > 4 \\ \text{By division,} & \underline{3 > 2} & \text{By division,} & \underline{3 > 2} \\ & 2\frac{1}{3} > 2 & & 1\frac{2}{3} < 2 \end{array}$$

1. Find one limit for x, if

$$4x - 3 > \frac{3x}{2} - \frac{3}{5}.$$

Multiply by 10, $40x - 30 > 15x - 6.$
Transpose, $25x > 24.$
Divide by 25, $x > \frac{24}{25}.$

2. Find the limits of x,

given $x - 4 > 2 - 3x,$ (1)
and $3x - 2 < x + 3.$ (2)

Transpose in (1), $4x > 6.$
Divide by 4, $x > 1\frac{1}{2}.$
Transpose in (2), $2x < 5.$
Divide by 2, $x < 2\frac{1}{2}.$

Therefore, the value of x lies between $1\frac{1}{2}$ and $2\frac{1}{2}$.

Exercise 80.

Find one limit for x, given:

1. $(x+1)^2 < x^2 + 3x - 5.$
2. $\frac{4x-2}{3} > \frac{3-5x}{7}.$
3. $x + 2b > 7x.$
4. $3x - 2 < \frac{x}{2} + 7\frac{1}{3}.$

5. Find the limiting values of x,

given $4x - 6 < 2x + 4,$
and $2x + 4 > 16 - 2x.$

6. If $a < b$, find the limiting values of x,

given $$\frac{ax}{5} + bx - ab > \frac{a^2}{5},$$
and $$\frac{bx}{7} - ax + ab < \frac{b^2}{7}.$$

7. Find the *integral* value of x,

given $\frac{1}{4}(x+2) + \frac{1}{3}x < \frac{1}{2}(x-4) + 3,$
and $\frac{1}{4}(x+2) + \frac{1}{3}x > \frac{1}{2}(x+1) + \frac{1}{3}.$

8. Twice a certain *integral* number increased by 7 is not greater than 19; and three times the number diminished by 5 is not less than 13. Find the number.

9. Twice the number of pupils in a certain class is less than 3 times the number minus 39; and 4 times the number plus 20 is greater than 5 times the number minus 21. Find the number of pupils in the class.

213. Theorem. *If a and b are unequal,* $a^2 + b^2 > 2ab$.

For $(a - b)^2$ must be positive, whatever the values of a and b.

That is, $(a - b)^2 > 0.$

Squaring, $a^2 - 2ab + b^2 > 0.$

Transposing $-2ab$, $a^2 + b^2 > 2ab.$

If a and b are positive and unequal, show that

$$a^3 + b^3 > a^2b + ab^2.$$

Now, $a^3 + b^3 > a^2b + ab^2,$

if (dividing each side by $a + b$)

$$a^2 - ab + b^2 > ab,$$

if (transposing $-ab$) $a^2 + b^2 > 2ab.$

But $a^2 + b^2 > 2ab.$ (Theorem)

Therefore, $a^3 + b^3 > a^2b + ab^2.$

EXERCISE 81.

If the letters are unequal and positive, show that:

1. $a^2 + 3b^2 > 2b(a + b).$
2. $(a^2 + b^2)(a^4 + b^4) > (a^3 + b^3)^2.$
3. $a^2b + a^2c + ab^2 + b^2c + ac^2 + bc^2 > 6abc.$
4. The sum of any fraction and its reciprocal $> 2.$
5. $ab + ac + bc < (a + b - c)^2 + (a + c - b)^2 + (b + c - a)^2.$
6. $(a^2 + b^2)(c^2 + d^2) > (ac + bd)^2.$
7. $\dfrac{a + b}{2} > \dfrac{2ab}{a + b}.$
8. $\dfrac{a}{b^2} + \dfrac{b}{a^2} > \dfrac{1}{b} + \dfrac{1}{a}.$

CHAPTER XV.

INVOLUTION AND EVOLUTION.

Involution.

214. The operation of raising an expression to any required *power* is called **Involution.**

215. **Index Law for Involution.** If m is a positive integer,

$$a^m = a \times a \times a \cdots\cdots \text{to } m \text{ factors.}$$

Consequently, if m and n are both positive integers,

$$\begin{aligned}(a^n)^m &= a^n \times a^n \times a^n \cdots\cdots \text{to } m \text{ factors}\\ &= a^{n+n+n \cdots\cdots \text{ to } m \text{ terms}}\\ &= a^{mn}.\end{aligned}$$ Hence,

Any required power of a given power of a number is found by multiplying the exponent of the given power by the exponent of the required power.

216. To find $(ab)^n$.

$$\begin{aligned}(ab)^n &= ab \times ab \cdots\cdots \text{to } n \text{ factors}\\ &= (a \times a \cdots\cdots \text{to } n \text{ factors})\,(b \times b \cdots\cdots \text{to } n \text{ factors})\\ &= a^n b^n.\end{aligned}$$

In like manner, $(abc)^n = a^n b^n c^n$; and so on. Hence,

Any required power of a product is found by taking the product of its factors each raised to the required power.

217. In the same way it may be shown that

Any required power of a fraction is found by taking the required power of the numerator and of the denominator.

218. From the *Law of Signs* in multiplication it is evident that all **even powers** of a number are **positive**; all **odd powers** of a number have the **same sign as the number itself.**

Hence, no *even* power of *any* number can be *negative*; and the even powers of two compound expressions that have the same terms with opposite signs are identical.

Thus, $$(b-a)^2=(a-b)^2.$$

219. Binomials. By actual multiplication we obtain,

$$(a+b)^2=a^2+2\,ab+b^2;$$
$$(a+b)^3=a^3+3\,a^2b+3\,ab^2+b^3;$$
$$(a+b)^4=a^4+4\,a^3b+6\,a^2b^2+4\,ab^3+b^4.$$

In these results it will be observed that:

1. The number of terms is greater by one than the exponent of the binomial.

2. In the first term the exponent of a is the same as the exponent of the binomial, and the exponent of a decreases by one in each succeeding term.

3. b appears in the second term with 1 for an exponent, and its exponent increases by 1 in each succeeding term.

4. The coefficient of the first term is 1.

5. The coefficient of the second term is the same as the exponent of the binomial.

6. The coefficient of each succeeding term is found from the next preceding term by multiplying the coefficient of that term by the exponent of a and dividing the product by a number greater by one than the exponent of b.

220. If b is negative, the terms in which the **odd** powers of b occur are **negative.** Thus,

1. $(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3.$
2. $(a - b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4.$

By the above rules any power of a binomial of the form $a \pm b$ may be written at once.

NOTE. The double sign ± is read *plus or minus;* and $a \pm b$ means $a + b$ or $a - b$.

221. The same method may be employed when the terms of a binomial have *coefficients* or *exponents.*

1. Find the third power of $5x^2 - 2y^3$.

Since $(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3,$

putting $5x^2$ for a, and $2y^3$ for b, we have

$$\begin{aligned}&(5x^2 - 2y^3)^3\\ &= (5x^2)^3 - 3(5x^2)^2(2y^3) + 3(5x^2)(2y^3)^2 - (2y^3)^3\\ &= 125x^6 - 150x^4y^3 + 60x^2y^6 - 8y^9\end{aligned}$$

2. Find the fourth power of $x^2 - \frac{1}{2}y$.

Since $(a - b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4,$

putting x^2 for a, and $\frac{1}{2}y$ for b, we have

$$\begin{aligned}&(x^2 - \tfrac{1}{2}y)^4\\ &= (x^2)^4 - 4(x^2)^3(\tfrac{1}{2}y) + 6(x^2)^2(\tfrac{1}{2}y)^2 - 4x^2(\tfrac{1}{2}y)^3 + (\tfrac{1}{2}y)^4\\ &= x^8 - 2x^6y + \tfrac{3}{2}x^4y^2 - \tfrac{1}{2}x^2y^3 + \tfrac{1}{16}y^4.\end{aligned}$$

222. In like manner, a *polynomial* of three or more terms may be raised to any power by enclosing its terms in parentheses, so as to give the expression the form of a binomial.

1. $$\begin{aligned}(a + b + c)^3 &= [a + (b + c)]^3\\ &= a^3 + 3a^2(b + c) + 3a(b + c)^2 + (b + c)^3\\ &= a^3 + 3a^2b + 3a^2c + 3ab^2 + 6abc\\ &\qquad + 3ac^2 + b^3 + 3b^2c + 3bc^2 + c^3.\end{aligned}$$

2. $(x^3 - 2x^2 + 3x + 4)^2$

$$= [(x^3 - 2x^2) + (3x + 4)]^2$$
$$= (x^3 - 2x^2)^2 + 2(x^3 - 2x^2)(3x + 4) + (3x + 4)^2$$
$$= x^6 - 4x^5 + 4x^4 + 6x^4 - 4x^3 - 16x^2 + 9x^2 + 24x + 16$$
$$= x^6 - 4x^5 + 10x^4 - 4x^3 - 7x^2 + 24x + 16.$$

Exercise 82.

Raise to the required power:

1. $(a^4)^3$.
2. $(a^2b^3)^5$.
3. $\left(\frac{2xy^2}{3ab^3}\right)^4$.
4. $(-5ab^2c^3)^4$.
5. $(-7x^2yz^3)^3$.
6. $\left(-\frac{3a^2b^3c^4}{2x^2y^5}\right)^5$.
7. $(-2x^2y^4)^6$.
8. $(-3a^2b^3x^4)^5$.
9. $\left(-\frac{3x^2y^3}{4z^3}\right)^4$.
10. $(x + 2)^5$.
11. $(x^2 - 2)^4$.
12. $(x + 3)^5$.
13. $(2x + 1)^6$.
14. $(2m^2 - 1)^5$.
15. $(2x + 3y)^5$.
16. $(2x - y)^6$.
17. $(xy - 2)^7$.
18. $(1 - x + x^2)^2$.
19. $(1 - 2x + 3x^2)^2$.
20. $(1 - a + a^2)^3$.
21. $(3 - 4x + 5x^2)^2$.

Evolution.

223. The operation of finding any required root of an expression is called **Evolution.** A root of an imperfect power cannot be found exactly.

Thus, the exact value of the square root of 2 can be written only as $\sqrt{2}$, and the exact value of the cube root of 4 can be written only as $\sqrt[3]{4}$. Approximate values of these expressions, however, can be found by annexing ciphers and extracting the root.

224. Index Law for Evolution. If m and n are positive integers, we have

$$(a^m)^n = a^{mn}. \qquad (\S\ 215)$$

Conversely, $$\sqrt[n]{a^{mn}} = a^{\frac{mn}{n}} = a^m.$$

Also, $$(ab)^n = a^n b^n. \qquad (\S\ 216)$$

Conversely, $$\sqrt[n]{a^n b^n} = \sqrt[n]{a^n} \times \sqrt[n]{b^n} = ab,$$

and $$\sqrt[n]{ab} = \sqrt[n]{a} \times \sqrt[n]{b}.$$

Thus, the *cube root* of a^6 is $a^{\frac{6}{3}} = a^2$; the *fourth root* of $81\,a^{12}$ is found by taking the fourth root of 81 and of a^{12}; and is $3\,a^3$. Hence,

225. To Find the Root of a Simple Expression,

Take the required root of the numerical coefficient, and divide the exponent of each letter by the index of the required root.

226. From the Law of Signs it is evident that:

1. Any **even root** of a **positive number** will have the double sign, ±.

2. There can be **no even root** of a **negative number.**

For $\sqrt{-x^2}$ is neither $+x$ nor $-x$; since the square of $+x = +x^2$, and the square of $-x = +x^2$.

The indicated even root of a negative number is called an **imaginary number.**

3. Any **odd root** of a number will have the **same sign as the number.**

If n is a positive integer, $$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}. \qquad (\S\ 217)$$

Conversely, $$\sqrt[n]{\frac{a^n}{b^n}} = \frac{\sqrt[n]{a^n}}{\sqrt[n]{b^n}} = \frac{a}{b}.$$ That is,

Any required root of a fraction is found by taking the required root of the numerator and of the denominator.

Thus, $\sqrt{\dfrac{16\,x^2}{81\,y^2}} = \pm\dfrac{4\,x}{9\,y}$; $\sqrt[3]{-27\,m^3n^6} = -3\,mn^2$;

$$\sqrt[4]{\frac{16\,x^8y^{12}}{81\,a^{16}}} = \pm\frac{2\,x^2y^3}{3\,a^4}.$$

227. If the root of a number expressed in figures is not readily detected, it may be found by resolving the number into its prime factors. Thus, to find the square root of 3,415,104:

$$\begin{array}{r|r}
2^3 & 3415104 \\ \hline
2^3 & 426888 \\ \hline
3^2 & 53361 \\ \hline
7 & 5929 \\ \hline
7 & 847 \\ \hline
11 & 121 \\ \hline
 & 11
\end{array}$$

$$\therefore \quad 3{,}415{,}104 = 2^6 \times 3^2 \times 7^2 \times 11^2.$$

$$\therefore \sqrt{3{,}415{,}104} = 2^3 \times 3 \times 7 \times 11 = 1848.$$

Exercise 83.

Simplify:

1. $\sqrt{4\,x^2y^4}$.
2. $\sqrt[3]{64\,x^9}$.
3. $\sqrt[4]{16\,x^8y^{12}}$.
4. $\sqrt[5]{-32\,a^{10}}$.
5. $\sqrt[3]{-27\,x^3}$.
6. $\sqrt{25\,a^4}$.
7. $\sqrt[3]{-8\,a^3b^6}$.
8. $\sqrt[6]{64\,x^{12}}$.
9. $\sqrt[3]{-216\,a^{12}}$.
10. $\sqrt[6]{729\,x^{18}}$.
11. $\sqrt[5]{243\,y^5z^{10}}$.
12. $\sqrt[3]{-1728\,d^3}$.
13. $\sqrt[3]{-343\,a^6}$.
14. $\sqrt[4]{81\,a^{24}}$.
15. $\sqrt[3]{512\,a^{12}b^{15}}$.
16. $\sqrt[3]{x^{3n}y^{12m}}$.
17. $\sqrt{\dfrac{9\,a^2b^6}{16\,x^4y^2}}$.
18. $\sqrt[3]{-\dfrac{8\,x^3y^6}{27\,z^9}}$.
19. $\sqrt[5]{-\dfrac{32\,a^{10}}{243\,x^{15}}}$.
20. $\sqrt[4]{\dfrac{16\,x^4}{81\,a^8b^{12}}}$.
21. $\sqrt[3]{\dfrac{125\,x^{21}}{216\,a^{24}}}$.

Square Roots of Compound Expressions.

228. Since the square of $a + b$ is $a^2 + 2ab + b^2$, the square root of $a^2 + 2ab + b^2$ is $a + b$.

It is required to find a method for extracting the square root, $a + b$, when $a^2 + 2ab + b^2$ is given:

$$\begin{array}{r|l}
 & a^2 + 2ab + b^2 \,\underline{|\, a + b} \\
 & a^2 \\ \hline
2a + b & 2ab + b^2 \\
 & \underline{2ab + b^2}
\end{array}$$

The first term, a, of the root is obviously the square root of the first term, a^2, of the expression.

If the a^2 is subtracted from the given expression, the remainder is $2ab + b^2$. Therefore, the second term, b, of the root is obtained when the first term of this remainder is divided by $2a$, that is, by *double the part of the root already found.* Also, since

$$2ab + b^2 = (2a + b)b,$$

the divisor *is completed by adding to the trial divisor the new term of the root.*

Find the square root of $25x^2 - 20x^3y + 4x^4y^2$.

$$\begin{array}{r|l}
 & 25x^2 - 20x^3y + 4x^4y^2 \,\underline{|\, 5x - 2x^2y} \\
 & 25x^2 \\ \hline
10x - 2x^2y & -20x^3y + 4x^4y^2 \\
 & \underline{-20x^3y + 4x^4y^2}
\end{array}$$

The expression is *arranged* according to the ascending powers of x.

The square root of the first term is $5x$, and $5x$ is placed at the right of the given expression, for the first term of the root.

The second term of the root, $-2x^2y$, is obtained by dividing $-20x^3y$ by $10x$, and this new term of the root is also annexed to the divisor, $10x$, to complete the divisor.

229. The same method will apply to longer expressions, if care is taken to obtain the *trial divisor* at each stage of the process, by *doubling the part of the root already found,* and to obtain the *complete divisor by annexing the new term of the root to the trial divisor.*

Find the square root of

$$1 + 10x^2 + 25x^4 + 16x^6 - 24x^5 - 20x^3 - 4x.$$

Arrange the expression in descending powers of x.

$$\begin{array}{r|l}
 & 16x^6 - 24x^5 + 25x^4 - 20x^3 + 10x^2 - 4x + 1 \;\underline{|\,4x^3 - 3x^2 + 2x - 1} \\
 & 16x^6 \\
8x^3 - 3x^2 & -24x^5 + 25x^4 \\
 & \underline{-24x^5 + 9x^4} \\
8x^3 - 6x^2 + 2x & 16x^4 - 20x^3 + 10x^2 \\
 & \underline{16x^4 - 12x^3 + 4x^2} \\
8x^3 - 6x^2 + 4x - 1 & -8x^3 + 6x^2 - 4x + 1 \\
 & \underline{-8x^3 + 6x^2 - 4x + 1}
\end{array}$$

It will be noticed that each successive trial divisor may be obtained by taking the preceding complete divisor with its *last term doubled.*

EXERCISE 84.

Find the square root of:

1. $x^4 - 8x^3 + 18x^2 - 8x + 1.$
2. $9a^4 - 6a^3 + 13a^2 - 4a + 4.$
3. $4x^4 - 12x^3y + 29x^2y^2 - 30xy^3 + 25y^4.$
4. $1 + 4x + 10x^2 + 12x^3 + 9x^4.$
5. $16 - 96x + 216x^2 - 216x^3 + 81x^4.$
6. $x^4 - 22x^3 + 95x^2 + 286x + 169.$
7. $4x^4 - 11x^2 + 25 - 12x^3 + 30x.$
8. $9x^4 + 49 - 12x^3 - 28x + 46x^2.$
9. $49x^4 + 126x^3 + 121 - 73x^2 - 198x.$
10. $16x^4 - 30x - 31x^2 + 24x^3 + 25.$
11. $x^4 - 2ax^3 + 3a^2x^2 - 2a^3x + a^4.$
12. $9x^4 - 18x + 1 + 87x^2 - 54x^3.$

230. If an expression contains powers and reciprocals of powers of the same letter, *the order of arrangement* in descending powers of the letter is as follows:

$$x^3,\ x^2,\ x,\ 1,\ \frac{1}{x},\ \frac{1}{x^2},\ \frac{1}{x^3}.$$

Find the square root of

$$\frac{9c^2}{x^2}+\frac{4x^2}{9c^2}-\frac{4x}{15c}+\frac{101}{25}-\frac{6c}{5x}.$$

Arrange in descending powers of x.

$$\begin{array}{r|l}
 & \frac{4x^2}{9c^2}-\frac{4x}{15c}+\frac{101}{25}-\frac{6c}{5x}+\frac{9c^2}{x^2}\ \Big|\ \underline{\frac{2x}{3c}-\frac{1}{5}+\frac{3c}{x}} \\
 & \frac{4x^2}{9c^2} \\
\frac{4x}{3c}-\frac{1}{5} & -\frac{4x}{15c}+\frac{101}{25} \\
 & \underline{-\frac{4x}{15c}+\frac{1}{25}} \\
\frac{4x}{3c}-\frac{2}{5}+\frac{3c}{x} & 4-\frac{6c}{5x}+\frac{9c^2}{x^2} \\
 & \underline{4-\frac{6c}{5x}+\frac{9c^2}{x^2}}
\end{array}$$

231. An approximate value of an imperfect square can be found to any required number of terms as follows:

Find to three terms the square root of x^2+px.

$$\begin{array}{r|l}
 & x^2+px\ \Big|\ \underline{x+\frac{p}{2}-\frac{p^2}{8x}} \\
 & x^2 \\
2x+\frac{p}{2} & px \\
 & \underline{px+\frac{p^2}{4}} \\
2x+p & -\frac{p^2}{4} \\
 & \underline{-\frac{p^2}{4}-\frac{p^3}{8x}}
\end{array}$$

Exercise 85.

Find the square root of:

1. $4x^4 + 4x^3 - \frac{1}{2}x + \frac{1}{16}$.

2. $\frac{4a^2}{b^2} + 8 + \frac{4b^2}{a^2}$.

3. $a^4 - 2a^3 + \frac{3a^2}{2} - \frac{a}{2} + \frac{1}{16}$.

4. $x^4 + \frac{2x^3}{3} + \frac{10x^2}{9} + \frac{x}{3} + \frac{1}{4}$.

5. $\frac{4x^2}{y^2} + \frac{3x}{y} + \frac{41}{16} + \frac{3y}{4x} + \frac{y^2}{4x^2}$.

6. $\frac{a^2}{4} - ax + \frac{3a}{2} + x^2 - 3x + \frac{9}{4}$.

7. $16x^4 + \frac{16}{3}x^2y + 8x^2 + \frac{4}{9}y^2 + \frac{4}{3}y + 1$.

8. $\frac{9x^4}{4} - \frac{3x^3}{2} + \frac{43x^2}{4} - \frac{7x}{2} + \frac{49}{4}$.

9. $4a^2 + \frac{9}{a^2} - \frac{6}{a} - 11 + 4a$.

Find to three terms the square root of:

10. $a^2 + b$.
11. $x^2 + \frac{1}{2}y$.
12. $1 + 2a$.
13. $1 + a$.
14. $1 - 2a$.
15. $4a^2 + 2b$.
16. $4x^2 + 3$.
17. $4 - 3a$.
18. $4a^2 - 1$.

232. Arithmetical Square Roots. In extracting the square root of an arithmetical number, the first step is to arrange the figures in *groups*.

Since $1 = 1^2$, $100 = 10^2$, $10,000 = 100^2$, and so on, the square root of a number between 1 and 100 lies between 1 and 10; of a number between 100 and 10,000 lies between

10 and 100. In other words, the square root of a number expressed by *one* or *two* figures is a number of *one* figure; of a number expressed by *three* or *four* figures is a number of *two* figures; and so on.

If, therefore, an integral square number is divided into groups of two figures each, from the right to the left, the number of figures in the root will be equal to the number of groups of figures. The last group to the left may have one figure or two figures.

Find the square root of 3249.

$$\begin{array}{rl} & 32\ 49\ (57 \\ & \underline{25} \\ 107) & 7\ 49 \\ & \underline{7\ 49} \end{array}$$

In this case, a in the typical form $a^2 + 2ab + b^2$ represents 5 *tens*, that is, 50, and b represents 7. The 25 subtracted is really 2500, that is, a^2, and the complete divisor $2a + b$ is $2 \times 50 + 7 = 107$.

233. The same method will apply to numbers of more than two groups of figures by considering a in the typical form to represent at each step *the part of the root already found.*

It must be observed that a represents *so many tens with respect to the next figure of the root.*

Find the square root of 5,322,249.

$$\begin{array}{rl} & 5\ 32\ 22\ 49\ (2307 \\ & \underline{4} \\ 43) & 1\ 32 \\ & \underline{1\ 29} \\ 4607) & 3\ 22\ 49 \\ & \underline{3\ 22\ 49} \end{array}$$

234. If the square root of a number has decimal places, the number itself will have *twice* as many. Thus, if 0.21 is the square root of some number, this number will be $(0.21)^2 = 0.21 \times 0.21 = 0.0441$; and if 0.111 is the root, the number will be $(0.111)^2 = 0.111 \times 0.111 = 0.012321$.

Therefore, the number of *decimal* places in every square decimal will be *even*, and the number of decimal places in the root will be *half* as many as in the given number itself.

Hence, if a given number contains a decimal, we divide the number into groups of two figures each, by beginning at the decimal point and marking toward the left for the integral number, and toward the right for the decimal. We must be careful to have the last group on the right of the decimal point contain *two* figures, annexing a cipher when necessary.

Find the square root of 41.2164; of 965.9664.

```
       41.21 64 (6.42
       36
  124) 5 21
       4 96
 1282) 25 64
       25 64
```

```
       9 65.96 64 (31.08
       9
   61) 65
       61
 6208) 4 96 64
       4 96 64
```

235. If a number contains an *odd* number of decimal places, or if any number gives a *remainder* when as many figures in the root have been obtained as the given number has groups, then its exact square root cannot be found. We may, however, approximate to its exact root as near as we please by annexing ciphers and continuing the operation.

The square root of a common fraction whose denominator is not a perfect square can be found approximately by reducing the fraction to a decimal and then extracting the root; or by reducing the fraction to an equivalent fraction whose denominator is a perfect square, and extracting the square root of both terms of the fraction. Thus,

$$\sqrt{\frac{5}{8}} = \sqrt{0.625} = 0.79057;$$

or
$$\sqrt{\frac{5}{8}} = \sqrt{\frac{10}{16}} = \frac{\sqrt{10}}{\sqrt{16}} = \frac{\sqrt{10}}{4} = \frac{3.16227}{4} = 0.79057$$

Find the square root of 3; of 357.357.

```
         3.(1.732.....               3 57.35 70 (18.903.....
         1                           1
   27) 2 00                    28) 2 57
       1 89                        2 24
  343) 11 00                  369) 33 35
       10 29                       33 21
 3462) 71 00                37803) 14 7000
        69 24                       11 3409
```

Exercise 86.

Find the square root of:

1. 289.
2. 1225.
3. 12,544.
4. 253,009.
5. 529,984.
6. 150.0625.
7. 118.1569.
8. 172.3969.
9. 5200.140544.
10. 1303.282201.
11. 640.343025.
12. 100.240144.
13. 316.021729.
14. 454.585041.
15. 5127.276025.

Find to four decimal places the square root of:

16. 10.
17. 3.
18. 5.
19. 0.5.
20. 0.7.
21. 0.9.
22. 0.607.
23. 0.521.
24. 0.687.
25. $\frac{2}{3}$.
26. $\frac{3}{4}$.
27. $\frac{4}{5}$.
28. $\frac{6}{7}$.
29. $\frac{5}{8}$.
30. $\frac{9}{11}$.

Cube Roots of Compound Expressions.

236. Since the cube of $a + b$ is $a^3 + 3a^2b + 3ab^2 + b^3$, the cube root of

$$a^3 + 3a^2b + 3ab^2 + b^3 \text{ is } a + b.$$

It is required to devise a method for extracting the cube root, $a + b$, when $a^3 + 3a^2b + 3ab^2 + b^3$ is given:

The first term, a, of the root is obviously the cube root of the first term, a^3, of the given expression.

$$
\begin{array}{l|l|l}
 & a^3 + 3a^2b + 3ab^2 + b^3 & \underline{a + b} \\
3a^2 & a^3 & \\
\underline{\qquad + 3ab + b^2} & 3a^2b + 3ab^2 + b^3 & \\
3a^2 + 3ab + b^2 & \underline{3a^2b + 3ab^2 + b^3} &
\end{array}
$$

If a^3 is subtracted, the remainder is $3a^2b + 3ab^2 + b^3$; therefore, the second term, b, of the root is obtained by dividing the first term of this remainder by *three times the square of* a.

Also, since $3a^2b + 3ab^2 + b^3 = (3a^2 + 3ab + b^2)b$, the *complete divisor* is obtained by adding $3ab + b^2$ to the *trial divisor* $3a^2$.

Find the cube root of $8x^3 + 36x^2y + 54xy^2 + 27y^3$.

$$
\begin{array}{rl|l|l}
 & & 8x^3 + 36x^2y + 54xy^2 + 27y^3 & \underline{2x + 3y} \\
 & 12x^2 & 8x^3 & \\
(6x + 3y)3y = & \underline{\qquad + 18xy + 9y^2} & 36x^2y + 54xy^2 + 27y^3 & \\
 & 12x^2 + 18xy + 9y^2 & \underline{36x^2y + 54xy^2 + 27y^3} &
\end{array}
$$

The cube root of the first term is $2x$, and $2x$ is therefore the first term of the root. $8x^3$, the cube of $2x$, is subtracted.

The second term of the root, $3y$, is obtained by dividing $36x^2y$ by $3(2x)^2 = 12x^2$, which corresponds to $3a^2$ in the typical form, and the divisor is completed by annexing to $12x^2$ the expression

$$\{3(2x) + 3y\}3y = 18xy + 9y^2.$$

237. The same method may be applied to longer expressions by considering a in the typical form $3a^2 + 3ab + b^2$ to represent at each stage of the process *the part of the root already found.* Thus, if the part of the root already found is $x + y$, then $3a^2$ of the typical form will be represented by $3(x + y)^2$; and if the third term of the root is $+ z$, the $3ab + b^2$ will be represented by $3(x + y)z + z^2$. So that the complete divisor, $3a^2 + 3ab + b^2$, will be represented by $3(x + y)^2 + 3(x + y)z + z^2$.

Find the cube root of $x^6 - 3x^5 + 5x^3 - 3x - 1$.

$$\begin{array}{rr|l}
 & & \underline{x^2 - x - 1} \\
 & & x^6 - 3x^5 + 5x^3 - 3x - 1 \\
 & 3x^4 \qquad\qquad\quad & x^6 \\
(3x^2 - x)(-x) = & \underline{-3x^3 + x^2} & -3x^5 + 5x^3 \\
 & 3x^4 - 3x^3 + x^2 & \underline{-3x^5 + 3x^4 - x^3} \\
 & & -3x^4 + 6x^3 - 3x - 1 \\
3(x^2 - x)^2 = & 3x^4 - 6x^3 + 3x^2 & \\
(3x^2 - 3x - 1)(-1) = & \underline{-3x^2 + 3x + 1} & \\
 & 3x^4 - 6x^3 \qquad + 3x + 1 & \underline{-3x^4 + 6x^3 - 3x - 1}
\end{array}$$

The root is placed above the given expression for convenience of arrangement on the page.

The first term of the root, x^2, is obtained by taking the cube root of the first term of the given expression; and the first trial divisor, $3x^4$, is obtained by taking three times the square of this term.

The first complete divisor is found by annexing to the trial divisor $(3x^2 - x)(-x)$, which expression corresponds to $(3a + b)b$ in the typical form.

The part of the root already found (a) is now represented by $x^2 - x$; therefore, $3a^2$ is represented by $3(x^2 - x)^2 = 3x^4 - 6x^3 + 3x^2$, the second trial divisor; and $(3a + b)b$ by $(3x^2 - 3x - 1)(-1)$; therefore, in the second complete divisor, $3a^2 + (3a + b)b$ is represented by $(3x^4 - 6x^3 + 3x^2) + (3x^2 - 3x - 1)(-1) = 3x^4 - 6x^3 + 3x + 1$.

EXERCISE 87.

Find the cube root of:

1. $a^3 + 3a^2x + 3ax^2 + x^3$.
2. $8 + 12x + 6x^2 + x^3$.
3. $x^6 - 3ax^5 + 5a^3x^3 - 3a^5x - a^6$.
4. $1 - 6x + 21x^2 - 44x^3 + 63x^4 - 54x^5 + 27x^6$.
5. $1 - 3x + 6x^2 - 7x^3 + 6x^4 - 3x^5 + x^6$.
6. $x^6 + 1 - 6x - 6x^5 + 15x^2 + 15x^4 - 20x^3$.

7. $64x^6 - 144x^5 + 8 - 36x + 102x^2 - 171x^3 + 204x^4.$

8. $27a^6 - 27a^5 - 18a^4 + 17a^3 + 6a^2 - 3a - 1.$

9. $8x^6 - 36x^5 + 66x^4 - 63x^3 + 33x^2 - 9x + 1.$

10. $27 + 108x + 90x^2 - 80x^3 - 60x^4 + 48x^5 - 8x^6.$

11. $a^3 - a^2b + \frac{ab^2}{3} - \frac{b^3}{27}.$

12. $\frac{x^9}{y^{15}} + \frac{3x^8}{y^{14}} - \frac{5x^6}{y^{12}} + \frac{3x^4}{y^{10}} - \frac{x^3}{y^9}.$

238. Arithmetical Cube Roots. In extracting the cube root of an arithmetical number, the first step is to arrange the figures in groups.

Since $1 = 1^3$, $1000 = 10^3$, $1,000,000 = 100^3$, and so on, it follows that the cube root of any number between 1 and 1000, that is, of any number which has *one, two,* or *three* figures, is a number of *one* figure; and that the cube root of any number between 1000 and 1,000,000, that is, of any number which has *four, five,* or *six* figures, is a number of *two* figures; and so on.

If, therefore, an integral cube number is divided into groups of three figures each, from right to left, the number of figures in the root will be equal to the number of groups. The last group to the left may have one, two, or three figures.

239. If the cube root of a number has decimal places, the number itself will have *three times* as many. Thus, if 0.11 is the cube root of a number, the number is $0.11 \times 0.11 \times 0.11 = 0.001331$. Hence, if a given number contains a decimal, we divide the number into groups of three figures each, by beginning at the decimal point and marking toward the left for the integral number, and toward the right for the decimal. We must be careful to have the last group on the right of the decimal point contain *three* figures, annexing ciphers when necessary.

240. Notice that if a denotes the first term, and b the second term of the root, the *first complete divisor* is

$$3\,a^2 + 3\,ab + b^2,$$

and the *second trial divisor* is $3\,(a+b)^2$, that is,

$$3\,a^2 + 6\,ab + 3\,b^2,$$

which may be obtained by adding to the preceding complete divisor *its second term and twice its third term.*

Extract the cube root of 5 to five places of decimals.

$$\begin{array}{rr|l}
 & & 5.000\,(1.70997 \\
 & & 1 \\
3 \times 10^2 = & 300 & 4000 \\
3\,(10 \times 7) = & 210 & \\
7^2 = & \underline{49} & \\
 & 559 & 3913 \\
 & \underline{259} & \quad 87\ 000\ 000 \\
3 \times 1700^2 = & 8670000 & \\
3\,(1700 \times 9) = & 45900 & \\
9^2 = & \underline{81} & \\
 & 8715981 & \quad 78\ 443\ 829 \\
 & \underline{45981} & \quad\ 8\ 556\ 1710 \\
3 \times 1709^2 = & 8762043 & \quad\ 7\ 885\ 8387 \\
 & & \qquad 670\ 33230 \\
 & & \qquad 613\ 34301
\end{array}$$

After the first two figures of the root are found, the next trial divisor is obtained by bringing down the sum of the 210 and 49 obtained in completing the preceding divisor; then adding the three numbers connected by the brace, and annexing two ciphers to the result.

The last two figures of the root are found by division. The rule in such cases is that two less than the number of figures already obtained may be found by division without error, the divisor being three times the square of the part of the root already found.

EXERCISE 88.

Find the cube root of:

1. 4913.
2. 42,875.
3. 1,404,928.
4. 127,263,527.
5. 385,828.352.
6. 1838.265625.

Find to four decimal places the cube root of:

7. 87.
8. 10.
9. 3.02.
10. 2.05.
11. 0.05.
12. 0.677.
13. $\frac{2}{3}$.
14. $\frac{3}{4}$.
15. $\frac{9}{11}$.
16. $\frac{1}{13}$.

241. Since the fourth power is the square of the square, and the sixth power the square of the cube, the *fourth root* is the *square root* of the *square root*, and the *sixth root* is the *cube root* of the *square root.* In like manner, the eighth, ninth, twelfth, ····· root may be found.

EXERCISE 89.

Find the fourth root of:

1. $81x^4 + 108x^3 + 54x^2 + 12x + 1$.
2. $16x^4 - 32ax^3 + 24a^2x^2 - 8a^3x + a^4$.
3. $1 + 4x + x^8 + 4x^7 + 10x^6 + 16x^3 + 10x^2 + 19x^4 + 16x^5$.

Find the sixth root of:

4. $1 + 6d + d^6 + 6d^5 + 15d^4 + 20d^3 + 15d^2$.
5. $729 - 1458x + 1215x^2 - 540x^3 + 135x^4 - 18x^5 + x^6$.
6. $1 - 18y + 135y^2 - 540y^3 + 1215y^4 - 1458y^5 + 729y^6$.

Find the eighth root of:

7. $1 - 8y + 28y^2 - 56y^3 + 70y^4 - 56y^5 + 28y^6 - 8y^7 + y^8$.

CHAPTER XVI.

THEORY OF EXPONENTS.

242. If n is a positive integer, we have defined a^n to mean the product obtained by taking a as a factor n times, and called a^n the nth *power of* a; we have also defined $\sqrt[n]{a}$ as a number which taken n times as a factor gives the product a, and called $\sqrt[n]{a}$ the nth *root of* a.

243. By this definition of a^n the exponent n denotes simply *repetitions* of a as a factor; and such expressions as $a^{\frac{2}{3}}$, a^{-3} have no meaning. It is found convenient, however, to extend the meaning of a^n to include fractional and negative values of n.

244. If we do not define the meaning of a^n when n is fractional or negative, but require that the meaning of a^n must in all cases be such that the fundamental index law shall always hold true, namely,

$$a^m \times a^n = a^{m+n},$$

we shall find that this condition alone will be sufficient to define the meaning of a^n for all cases.

245. Meaning of a Zero Exponent. By the index law,

$$a^0 \times a^n = a^{0+n} = a^n.$$

Divide by a^n, $\qquad a^0 = \dfrac{a^n}{a^n} = 1.$

Therefore, the *zero* power of any number is equal to *unity*.

246. Meaning of a Fractional Exponent. By the index law,

$$a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^{\frac{2}{2}} = a;$$
$$a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}} = a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = a^{\frac{3}{3}} = a;$$
$$a^{\frac{3}{4}} \times a^{\frac{3}{4}} \times a^{\frac{3}{4}} \times a^{\frac{3}{4}} = a^{\frac{3}{4}+\frac{3}{4}+\frac{3}{4}+\frac{3}{4}} = a^{\frac{12}{4}} = a^3;$$
$$a^{\frac{1}{n}} \times a^{\frac{1}{n}} \cdots\cdots \text{to } n \text{ factors} = a^{\frac{1}{n}+\frac{1}{n} \cdots\cdots \text{to } n \text{ terms}} = a^{\frac{n}{n}} = a;$$
$$a^{\frac{m}{n}} \times a^{\frac{m}{n}} \cdots\cdots \text{to } n \text{ factors} = a^{\frac{m}{n}+\frac{m}{n} \cdots\cdots \text{to } n \text{ terms}} = a^{\frac{mn}{n}} = a^m;$$

provided m and n are positive integers.

That is,
$$a^{\frac{1}{2}} = \sqrt{a};\ a^{\frac{1}{3}} = \sqrt[3]{a};$$
$$a^{\frac{3}{4}} = \sqrt[4]{a^3};\ a^{\frac{m}{n}} = \sqrt[n]{a^m}.$$

The meaning, therefore, of $a^{\frac{m}{n}}$, where m and n are positive integers, is the nth root of the mth power of a. Hence,

The numerator of a fractional exponent indicates a power and the denominator a root.

247. Meaning of a Negative Exponent. By the index law, if n is a positive integer,

$$a^n \times a^{-n} = a^{n-n} = a^0.$$

But
$$a^0 = 1. \qquad (\S\ 245)$$
$$\therefore a^n \times a^{-n} = 1.$$

That is, a^n and a^{-n} are *reciprocals* of each other (§ 169),

so that $a^{-n} = \dfrac{1}{a^n}$, and $a^n = \dfrac{1}{a^{-n}}$.

248. Hence, we can change any *factor* from the numerator of a fraction to the denominator, or from the denominator to the numerator, *provided we change the sign of its exponent.*

Thus, $\dfrac{ab^2}{c^3d^3}$ may be written $ab^2c^{-3}d^{-3}$, or $\dfrac{1}{a^{-1}b^{-2}c^3d^3}$.

249. We have now assigned definite meanings to fractional exponents and negative exponents, by assuming that the *index law for multiplication*, $a^m \times a^n = a^{m+n}$, is true for *all values of the exponents m and n.*

It remains to show that the index laws established for division, involution, and evolution apply to *fractional* and *negative* exponents.

250. Index Law of Division for all Values of m and n. To *divide* by a number *is to multiply the dividend by the reciprocal of the divisor.*

Therefore, for all values of m and n,

$$\frac{a^m}{a^n} = a^m \times \frac{1}{a^n} = a^m \times a^{-n} \qquad (\S\ 247)$$
$$= a^{m-n}.$$

251. Index Law of Involution and Evolution for all Values of m and n.

To prove $(a^m)^n = a^{mn}$ *for all values of m and n.*

CASE 1. Let m have *any* value, and let n be a *positive integer.*

Then,
$$(a^m)^n = a^m \times a^m \times a^m \cdots\cdots \text{ to } n \text{ factors}$$
$$= a^{m+m+m \cdots\cdots \text{ to } n \text{ terms}}$$
$$= a^{mn}.$$

CASE 2. Let m have *any* value, and $n = \frac{p}{q}$, p and q being positive integers.

Then,
$$(a^m)^{\frac{p}{q}} = \sqrt[q]{(a^m)^p} = \sqrt[q]{a^{mp}}$$
$$= a^{\frac{mp}{q}}.$$

CASE 3. Let m have *any* value, and $n = -r$, r being a positive integer or a positive fraction.

Then, $$(a^m)^{-r} = \frac{1}{(a^m)^r} = \frac{1}{a^{mr}} = a^{-mr}. \qquad (\S\ 247)$$

Therefore, $(a^m)^n = a^{mn}$ for all values of m and n.

252. *To prove $(ab)^n = a^n b^n$ for any value of n.*

CASE 1. Let n be a positive integer.

Then, $$\begin{aligned}(ab)^n &= ab \times ab \times ab \cdots\cdots \text{ to } n \text{ factors}\\ &= (a \times a \cdots\cdots \text{to } n \text{ factors})(b \times b \cdots\cdots \text{to } n \text{ factors})\\ &= a^n b^n.\end{aligned}$$

CASE 2. Let $n = \frac{p}{q}$, p and q being positive integers.

Then, by Case 1, § 251, since q is a positive integer, the qth power of $$\begin{aligned}(ab)^{\frac{p}{q}} &= (ab)^{\frac{p}{q}} \times (ab)^{\frac{p}{q}} \cdots\cdots \text{ to } q \text{ factors}\\ &= (ab)^{\frac{p}{q}+\frac{p}{q}\cdots\cdots \text{ to } q \text{ terms}}\\ &= (ab)^p = a^p b^p. \qquad \text{(By Case 1)}\end{aligned}$$

Also by Case 1, § 251, since q is a positive integer, the qth power of

$$\begin{aligned}a^{\frac{p}{q}} b^{\frac{p}{q}} &= (a^{\frac{p}{q}} \times a^{\frac{p}{q}} \cdots\cdots \text{ to } q \text{ factors})\,(b^{\frac{p}{q}} \times b^{\frac{p}{q}} \cdots\cdots \text{ to } q \text{ factors})\\ &= a^p b^p.\end{aligned}$$

But the qth power of $(ab)^{\frac{p}{q}} = (ab)^p = a^p b^p$.

That is, $$\left[(ab)^{\frac{p}{q}}\right]^q = \left[a^{\frac{p}{q}} b^{\frac{p}{q}}\right]^q.$$

Extracting the qth root of each member, we have

$$(ab)^{\frac{p}{q}} = a^{\frac{p}{q}} b^{\frac{p}{q}}.$$

CASE 3. Let $n = -r$, r being a positive integer or fraction.

Then, $$(ab)^{-r} = \frac{1}{(ab)^r} = \frac{1}{a^r b^r} = a^{-r} b^{-r}.$$

EXAMPLES.

1. $27^{-\frac{1}{3}} = \dfrac{1}{27^{\frac{1}{3}}} = \dfrac{1}{\sqrt[3]{27}} = \dfrac{1}{3}.$

2. $16^{\frac{3}{4}} = (\sqrt[4]{16})^3 = (\pm 2)^3 = \pm 8.$

3. $a^{\frac{3}{4}} \times a^{-\frac{2}{3}} = a^{\frac{3}{4}-\frac{2}{3}} = a^{\frac{1}{12}} = \sqrt[12]{a}.$

4. $a^{\frac{5}{6}} \times a^{\frac{1}{2}} \times a^{-\frac{4}{3}} = a^{\frac{5}{6}+\frac{1}{2}-\frac{4}{3}} = a^0 = 1.$

5. $(a^{-\frac{1}{3}})^6 = a^{-\frac{1}{3}\times 6} = a^{-2} = \dfrac{1}{a^2}.$

6. $\dfrac{a^{-3}}{a^{-5}} = a^{-3-(-5)} = a^{-3+5} = a^2.$

7. $\sqrt[3]{a^2b^{-3}c^{-4}d} = a^{\frac{2}{3}}b^{-1}c^{-\frac{4}{3}}d^{\frac{1}{3}}.$

8. $(4\,a^{-\frac{2}{3}})^{-\frac{3}{2}} = \dfrac{1}{(4\,a^{-\frac{2}{3}})^{\frac{3}{2}}} = \dfrac{1}{4^{\frac{3}{2}}a^{-\frac{2}{3}\times\frac{3}{2}}} = \dfrac{1}{8\,a^{-1}} = \dfrac{a}{8}.$

9. $\left(\dfrac{16\,a^{-4}}{81\,b^3}\right)^{-\frac{3}{4}} = \left(\dfrac{81\,b^3}{16\,a^{-4}}\right)^{\frac{3}{4}} = \dfrac{27\,b^{\frac{9}{4}}}{8\,a^{-3}} = \dfrac{27\,a^3b^{\frac{9}{4}}}{8}.$

10. $(3^{\frac{3}{8}}a^{-3})^{-\frac{2}{3}} = \dfrac{1}{(3^{\frac{3}{8}}a^{-3})^{\frac{2}{3}}} = \dfrac{1}{3^{\frac{2}{3}\times\frac{3}{8}}a^{-2}} = \dfrac{a^2}{3^{\frac{1}{4}}} = \dfrac{a^2}{\sqrt[4]{3}}.$

Exercise 90.

Express with fractional exponents:

1. $\sqrt[3]{a^2}.$ 3. $\sqrt[5]{a^7}.$ 5. $\sqrt[4]{a^3}.$ 7. $\sqrt{a} + \sqrt[3]{x} + \sqrt[4]{16\,b^2}.$

2. $\sqrt{a^3}.$ 4. $\sqrt[3]{-8}.$ 6. $\sqrt[3]{a^5}.$ 8. $\sqrt[5]{a^2x^5} + \sqrt[3]{a^8c}.$

Express with radical signs:

9. $a^{\frac{3}{4}}$. 11. $a^{\frac{1}{2}}b^{\frac{1}{3}}$. 13. $x^{\frac{1}{3}}y^{-\frac{1}{3}}$. 15. $a^{\frac{1}{2}}-x^{\frac{1}{3}}c^{\frac{1}{3}}$.

10. $c^{\frac{2}{3}}$. 12. $a^{\frac{1}{2}}b^{\frac{2}{3}}$. 14. $3x^{\frac{2}{3}}y^{-\frac{3}{4}}$. 16. $a^{\frac{2}{3}}+x^{\frac{2}{3}}c^{\frac{2}{3}}$.

Express with positive exponents:

17. a^{-3}. 19. $3x^{-2}y^{3}$. 21. $4x^{-3}y^{-2}$. 23. $\frac{2a^{-1}x^{2}}{3^{-2}b^{2}y^{-5}}$.

18. $a^{-\frac{2}{3}}$. 20. $4xy^{-7}$. 22. $3a^{-4}b^{\frac{3}{2}}$.

Write without denominators:

24. $\frac{3y^{2}z^{3}}{x^{-5}}$. 26. $\frac{abc}{a^{-2}bc^{-2}d}$. 28. $\frac{x^{-2}y^{\frac{2}{3}}}{a^{-2}b^{-\frac{6}{5}}}$.

25. $\frac{x^{2}z}{x^{-3}z^{-2}}$. 27. $\frac{a^{-1}b^{-2}c^{-3}}{a^{\ 2}b^{-2}c^{-4}}$. 29. $\frac{a^{-4}b^{-5}c^{-6}}{a^{-7}b^{-5}c^{-8}}$.

Find the value of:

30. $8^{\frac{5}{3}}$. 35. $(-27)^{\frac{5}{3}}$. 40. $(\frac{1}{125})^{-\frac{2}{3}}\times(\frac{1}{36})^{\frac{1}{2}}$

31. $16^{-\frac{5}{4}}$. 36. $(-27)^{\frac{4}{3}}\times 25^{\frac{5}{2}}$. 41. $a^{-\frac{3}{4}}b^{\frac{2}{3}}\times a^{\frac{1}{4}}b^{\frac{1}{3}}$.

32. $27^{\frac{2}{3}}$. 37. $81^{-\frac{3}{2}}\times 16^{\frac{3}{4}}$. 42. $(a^{-\frac{2}{3}}b^{3})^{-\frac{3}{2}}$.

33. $(-8)^{-\frac{2}{3}}$. 38. $8^{\frac{2}{3}}\times 4^{-\frac{1}{2}}$. 43. $(a^{-\frac{1}{2}}b^{-1})^{-2}$.

34. $36^{\frac{3}{2}}$. 39. $(\frac{1}{25})^{\frac{1}{2}}\times 16^{-\frac{3}{4}}$. 44. $(x^{\frac{2}{3}}y^{-\frac{1}{2}}z^{\frac{1}{5}})^{-\frac{2}{5}}$.

If $a=4$, $b=2$, $c=1$, find the numerical value of:

45. $a^{\frac{1}{2}}b^{-1}$. 47. $a^{-\frac{1}{2}}b^{2}$. 49. $3(ab)^{\frac{1}{3}}$. 51. $(ab^{2}c)^{\frac{1}{4}}$.

46. ab^{-2}. 48. $a^{-\frac{1}{2}}c^{-\frac{2}{3}}$. 50. $2(ab)^{-\frac{1}{3}}$. 52. $(ab^{2}c)^{-\frac{1}{4}}$.

Multiplication.

Multiply $x^{\frac{1}{2}} + x^{\frac{1}{4}}y^{\frac{1}{4}} + y^{\frac{1}{2}}$ by $x^{\frac{1}{2}} - x^{\frac{1}{4}}y^{\frac{1}{4}} + y^{\frac{1}{2}}$.

$$\begin{array}{llll}
x^{\frac{1}{2}} + x^{\frac{1}{4}}y^{\frac{1}{4}} + y^{\frac{1}{2}} & & & \\
x^{\frac{1}{2}} - x^{\frac{1}{4}}y^{\frac{1}{4}} + y^{\frac{1}{2}} & & & \\
\hline
x & + x^{\frac{3}{4}}y^{\frac{1}{4}} & + x^{\frac{1}{2}}y^{\frac{1}{2}} & \\
 & - x^{\frac{3}{4}}y^{\frac{1}{4}} & - x^{\frac{1}{2}}y^{\frac{1}{2}} & - x^{\frac{1}{4}}y^{\frac{3}{4}} \\
 & & + x^{\frac{1}{2}}y^{\frac{1}{2}} & + x^{\frac{1}{4}}y^{\frac{3}{4}} + y \\
\hline
x & & + x^{\frac{1}{2}}y^{\frac{1}{2}} & + y
\end{array}$$

Exercise 91.

Multiply:

1. $a^{\frac{1}{2}} + b^{\frac{1}{2}}$ by $a^{\frac{1}{2}} - b^{\frac{1}{2}}$.
2. $a^{\frac{1}{2}} + b^{\frac{1}{4}}$ by $a^{\frac{1}{4}} + b^{\frac{1}{2}}$.
3. $a^{\frac{1}{2}} - b^{\frac{1}{2}}$ by $a^{\frac{1}{2}} - b^{\frac{1}{2}}$.
4. $x^{\frac{3}{2}} + 2x$ by $x^{\frac{3}{2}} - 2x$.
5. $x^{-2} + x^{-1}y^{-1} + y^{-2}$ by $x^{-2} - x^{-1}y^{-1} + y^{-2}$.
6. $x^{\frac{2}{3}} - x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}}$ by $x^{\frac{1}{3}} + y^{\frac{1}{3}}$.
7. $x^{\frac{2}{3}} + x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}}$ by $x^{\frac{1}{3}} - y^{\frac{1}{3}}$.
8. $1 + b^{-1} + b^{-2}$ by $1 - b^{-1} + b^{-2}$.
9. $a^{\frac{2}{3}}b^{-\frac{1}{2}} + 2a^{\frac{1}{3}} - 3b^{\frac{1}{2}}$ by $2b^{-\frac{1}{2}} - 4a^{-\frac{1}{3}} - 6a^{-\frac{2}{3}}b^{\frac{1}{2}}$.

Division.

Divide $\sqrt[3]{x^2} + \sqrt[3]{x} - 12$ by $\sqrt[3]{x} - 3$.

$$\begin{array}{l|l}
x^{\frac{2}{3}} + x^{\frac{1}{3}} - 12 & x^{\frac{1}{3}} - 3 \\
\cline{2-2}
x^{\frac{2}{3}} - 3x^{\frac{1}{3}} & x^{\frac{1}{3}} + 4 \\
\cline{1-1}
\quad + 4x^{\frac{1}{3}} - 12 & \\
\quad + 4x^{\frac{1}{3}} - 12 & \\
\cline{1-1}
\end{array}$$

Here $x^{\frac{2}{3}} \div x^{\frac{1}{3}} = x^{\frac{2}{3} - \frac{1}{3}} = x^{\frac{1}{3}}$; $4x^{8} \div x^{8} = 4x^{8-8} = 4x^{0} = 4$.

EXERCISE 92.

Divide:

1. $a - b$ by $a^{\frac{1}{3}} - b^{\frac{1}{3}}$.
2. $a + b$ by $a^{\frac{1}{3}} + b^{\frac{1}{3}}$.
3. $a - b$ by $a^{\frac{1}{4}} - b^{\frac{1}{4}}$.
4. $a + b$ by $a^{\frac{1}{5}} + b^{\frac{1}{5}}$.
5. $x - 3x^{\frac{2}{3}} + 3x^{\frac{1}{3}} - 1$ by $x^{\frac{1}{3}} - 1$.
6. $x + x^{\frac{1}{2}}y^{\frac{1}{2}} + y$ by $x^{\frac{1}{2}} - x^{\frac{1}{4}}y^{\frac{1}{4}} + y^{\frac{1}{2}}$.
7. $x^{\frac{2}{3}} - 4x^{\frac{1}{3}} + 1 + 6x^{-\frac{1}{3}}$ by $x^{\frac{1}{3}} - 2$.
8. $9x - 12x^{\frac{1}{2}} - 2 + 4x^{-\frac{1}{2}} + x^{-1}$ by $3x^{\frac{1}{2}} - 2 - x^{-\frac{1}{2}}$.
9. $2x^{-2} + 6x^{-1}y^{-1} - 16x^2y^{-4}$ by $2x + 2x^2y^{-1} + 4x^3y^{-2}$.
10. $x + y + z - 3x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}}$ by $x^{\frac{1}{3}} + y^{\frac{1}{3}} + z^{\frac{1}{3}}$.

Square Root.

Find the square root of

$$9x^{-4} - 18x^{-3}y^{\frac{1}{2}} + 15x^{-2}y - 6x^{-1}y^{\frac{3}{2}} + y^2.$$

$$\begin{array}{r|l|l}
 & 9x^{-4} - 18x^{-3}y^{\frac{1}{2}} + 15x^{-2}y - 6x^{-1}y^{\frac{3}{2}} + y^2 & 3x^{-2} - 3x^{-1}y^{\frac{1}{2}} + y \\
 & 9x^{-4} & \\
6x^{-2} - 3x^{-1}y^{\frac{1}{2}} & -18x^{-3}y^{\frac{1}{2}} + 15x^{-2}y & \\
 & -18x^{-3}y^{\frac{1}{2}} + 9x^{-2}y & \\
6x^{-2} - 6x^{-1}y^{\frac{1}{2}} + y & 6x^{-2}y - 6x^{-1}y^{\frac{3}{2}} + y^2 & \\
 & 6x^{-2}y - 6x^{-1}y^{\frac{3}{2}} + y^2 &
\end{array}$$

EXERCISE 93.

Find the square root of:

1. $x^{\frac{2}{3}} + 2x^{\frac{1}{3}} + 1$.
2. $4a^{\frac{2}{3}} - 4a^{\frac{1}{3}}b^{\frac{1}{4}} + b^{\frac{1}{2}}$.
3. $x^{\frac{2}{3}} - 4x^{\frac{1}{3}} + 4$.
4. $4a^{-2} + 4a^{-1} + 1$.
5. $9a - 12a^{\frac{1}{2}} + 10 - 4a^{-\frac{1}{2}} + a^{-1}$.
6. $m^2 + 2m - 1 - 2m^{-1} + m^{-2}$.
7. $49x^{\frac{4}{3}} - 28x + 18x^{\frac{2}{3}} - 4x^{\frac{1}{3}} + 1$.

CHAPTER XVII.

RADICAL EXPRESSIONS.

253. A **radical expression** is an expression affected by the radical sign; as $\sqrt{a}$, $\sqrt[6]{9}$, $\sqrt[3]{a^2}$, $\sqrt[4]{a+b}$, $\sqrt[5]{32}$.

254. An indicated root that cannot be exactly obtained is called a **surd.** An indicated root that can be exactly obtained is said to have the *form* of a surd.

The required root shows the **order** of a surd; and surds are named quadratic, cubic, biquadratic, according as the second, third, or fourth roots are required.

The product of a rational factor and a surd factor is called a **mixed surd**; as $3\sqrt{2}$, $b\sqrt{a}$. The rational factor of a mixed surd is called the **coefficient** of the surd.

When there is no rational factor outside of the radical sign, that is, when the coefficient is 1, the surd is said to be **entire**; as $\sqrt{2}$, $\sqrt{a}$.

255. A surd is in its **simplest form** when the expression under the radical sign is *integral and as small as possible.*

Surds are said to be **similar** if they have the same surd factor when reduced to the simplest form.

Note. In operations with surds, arithmetical numbers contained in the surds should be expressed in their prime factors.

Reduction of Radicals.

256. To reduce a radical is to change its *form* without changing its *value.*

CASE 1.

257. When the radical is a perfect power and has for an exponent a factor of the index of the root.

1. $\sqrt[4]{a^2} = a^{\frac{2}{4}} = a^{\frac{1}{2}} = \sqrt{a}.$

2. $\sqrt[4]{36\,a^2b^2} = \sqrt[4]{(6\,ab)^2} = (6\,ab)^{\frac{2}{4}} = (6\,ab)^{\frac{1}{2}} = \sqrt{6\,ab}.$

3. $\sqrt[6]{25\,a^4b^2c^8} = \sqrt[6]{(5\,a^2bc^4)^2} = (5\,a^2bc^4)^{\frac{2}{6}} = (5\,a^2bc^4)^{\frac{1}{3}}$
$= \sqrt[3]{5\,a^2bc^4}.$

We have, therefore, the following rule:

Divide the exponent of the power by the index of the root.

EXERCISE 94.

Simplify:

1. $\sqrt[4]{25}.$
2. $\sqrt[8]{16}.$
3. $\sqrt[6]{27}.$
4. $\sqrt[4]{49}.$
5. $\sqrt[6]{64}.$
6. $\sqrt[6]{a^2b^2}.$
7. $\sqrt[4]{a^2b^2}.$
8. $\sqrt[8]{a^4b^4}.$
9. $\sqrt[6]{27\,a^3b^6}.$
10. $\sqrt[8]{16\,a^4b^4}.$
11. $\sqrt[6]{\dfrac{25\,a^2}{64\,b^2}}.$
12. $\sqrt[4]{\dfrac{16\,x^2}{(x-3)^2}}.$
13. $\sqrt[6]{\dfrac{a^3b^6}{8\,x^3y^3}}.$

CASE 2.

258. When the radical is the product of two factors, one of which is a perfect power of the same degree as the radical.

Since $\sqrt[n]{a^nb} = \sqrt[n]{a^n} \times \sqrt[n]{b} = a\sqrt[n]{b}$ (§ 224), we have

1. $\sqrt{a^2b} = \sqrt{a^2} \times \sqrt{b} = a\sqrt{b};$

2. $\sqrt[3]{108} = \sqrt[3]{27 \times 4} = \sqrt[3]{27} \times \sqrt[3]{4} = 3\sqrt[3]{4};$

3. $4\sqrt{72\,a^2b^3} = 4\sqrt{36\,a^2b^2 \times 2\,b} = 4\sqrt{36\,a^2b^2} \times \sqrt{2\,b}$
$$= 4 \times 6\,ab\sqrt{2\,b} = 24\,ab\sqrt{2\,b};$$

4. $2\sqrt[3]{54\,a^4b} = 2\sqrt[3]{27\,a^3 \times 2\,ab} = 2\sqrt[3]{27\,a^3} \times \sqrt[3]{2\,ab}$
$$= 2 \times 3\,a\sqrt[3]{2\,ab} = 6\,a\sqrt[3]{2\,ab}.$$

We have, therefore, the following rule:

Resolve the radical into two factors, one of which is the greatest perfect power of the same degree as the radical.

Remove this factor from under the radical sign, extract the required root, and multiply the coefficient of the surd by the root obtained.

Exercise 95.

Simplify:

1. $\sqrt{28}$.
2. $\sqrt{72}$.
3. $\sqrt[3]{72}$.
4. $\sqrt[3]{500}$.
5. $\sqrt[3]{432}$.
6. $\sqrt[3]{192}$.
7. $\sqrt[5]{128}$.
8. $\sqrt[4]{243}$.
9. $\sqrt[4]{176}$.
10. $\sqrt{405}$.
11. $2\sqrt[4]{112}$.
12. $3\sqrt[3]{864}$.
13. $7\sqrt[4]{144}$.
14. $8\sqrt{m^2n}$.
15. $3\sqrt[4]{b^8a^3}$.
16. $2\sqrt[5]{a^{18}c^4}$.
17. $11\sqrt[6]{a^{12}b^{15}}$.
18. $7\sqrt[3]{8\,a^3b}$.
19. $6\sqrt[3]{27\,m^2n^3}$.
20. $4\sqrt[4]{x^7y^8}$.
21. $\sqrt[3]{1029}$.
22. $\sqrt[3]{-2187}$.
23. $\sqrt[4]{1250}$.
24. $4\sqrt[3]{648}$.
25. $\sqrt[3]{\dfrac{64\,x^6y}{27\,m^3n^3}}$.
26. $\sqrt{\dfrac{4\,a^3b}{9}}$.
27. $\sqrt[3]{\dfrac{125\,x^3}{216\,y^3}}$.
28. $2\sqrt[5]{\dfrac{m^6n}{243}}$.
29. $\sqrt[4]{\dfrac{x^5y^7}{1296}}$.
30. $\sqrt[3]{\dfrac{(x-y)^3z^3}{512}}$.
31. $\dfrac{3\,ab}{2\,c}\sqrt{\dfrac{20\,c^2}{9\,a^2b^2}}$.

CASE 3.

259. **When the radical expression is a fraction, the denominator of which is not a perfect power of the same degree as the radical.**

1. $\sqrt{\frac{5}{8}} = \sqrt{\frac{10}{16}} = \sqrt{10 \times \frac{1}{16}} = \frac{1}{4}\sqrt{10}.$

2. $\sqrt{\frac{7}{12}} = \sqrt{\frac{7}{4 \times 3}} = \sqrt{\frac{7 \times 3}{4 \times 9}} = \sqrt{21 \times \frac{1}{36}} = \frac{1}{6}\sqrt{21}.$

3. $\sqrt[3]{\frac{5}{18}} = \sqrt[3]{\frac{5}{9 \times 2}} = \sqrt[3]{\frac{5 \times 3 \times 4}{27 \times 8}} = \sqrt[3]{60 \times \frac{1}{27 \times 8}}$

$$= \frac{1}{3 \times 2}\sqrt[3]{60} = \frac{1}{6}\sqrt[3]{60}.$$

We have, therefore, the following rule:

Multiply both terms of the fraction by a number that will make the denominator a perfect power of the same degree as the radical; and then proceed as in Case 2.

EXERCISE 96.

Simplify:

1. $2\sqrt{\frac{1}{2}}$.
2. $\frac{3}{4}\sqrt{\frac{2}{3}}$.
3. $\frac{1}{4}\sqrt{\frac{1}{5}}$.
4. $7\sqrt{\frac{4}{5}}$.
5. $\sqrt[4]{\frac{25}{16}}$.
6. $3\sqrt{\frac{9}{80}}$.
7. $\sqrt[3]{\frac{8}{3}}$.
8. $\sqrt[3]{\frac{9}{8}}$.
9. $\sqrt[3]{\frac{24}{343}}$.
10. $2\sqrt[3]{\frac{5}{3}}$.
11. $3\sqrt[5]{\frac{2}{81}}$.
12. $2\sqrt[5]{\frac{3}{16}}$.
13. $\sqrt{\frac{a^4c}{b^3}}$.
14. $\sqrt[4]{\frac{b^4}{a^3}}$.
15. $\sqrt[3]{\frac{ax^4}{b^2}}$.
16. $\sqrt[3]{\frac{7a}{125x}}$.
17. $\sqrt{\frac{a^2cy^2}{bd^2}}$.
18. $2\sqrt[3]{\frac{2a^3b^2c}{3x^2yz^3}}$.

CASE 4.

260. To reduce a mixed surd to an entire surd.

Since $a\sqrt[n]{b} = \sqrt[n]{a^n} \times \sqrt[n]{b} = \sqrt[n]{a^nb}$, we have

1. $3\sqrt{5} = \sqrt{3^2 \times 5} = \sqrt{9 \times 5} = \sqrt{45}$;
2. $a^2b\sqrt{bc} = \sqrt{(a^2b)^2 \times bc} = \sqrt{a^4b^2 \times bc} = \sqrt{a^4b^3c}$;
3. $2x\sqrt[3]{xy} = \sqrt[3]{(2x)^3 \times xy} = \sqrt[3]{8x^3 \times xy} = \sqrt[3]{8x^4y}$;
4. $3y^2\sqrt[4]{x^3} = \sqrt[4]{(3y^2)^4 \times x^3} = \sqrt[4]{81y^8x^3}$.

We have, therefore, the following rule:

Raise the coefficient to a power of the same degree as the radical, multiply this power by the given surd factor, and indicate the required root of the product.

EXERCISE 97.

Express as an entire surd:

1. $5\sqrt{5}$.
2. $3\sqrt{11}$.
3. $3\sqrt[3]{3}$.
4. $2\sqrt[3]{4}$.
5. $2\sqrt[4]{3}$.
6. $3\sqrt[5]{2}$.
7. $2\sqrt[6]{2}$.
8. $2\sqrt[4]{4}$.
9. $-2\sqrt[3]{y}$.
10. $-3\sqrt{y^3}$.
11. $-m\sqrt[6]{10}$.
12. $-2\sqrt[4]{x}$.
13. $\frac{1}{2}\sqrt{a}$.
14. $-\frac{2}{3}\sqrt[3]{a^2}$.
15. $\frac{3}{4}\sqrt{m^3}$.
16. $-\frac{1}{2}\sqrt[4]{m^7}$

CASE 5.

261. To reduce radicals to a common index.

Reduce $\sqrt{2}$ and $\sqrt[3]{3}$ to a common index.

$$\sqrt{2} = 2^{\frac{1}{2}} = 2^{\frac{3}{6}} = \sqrt[6]{2^3} = \sqrt[6]{8}.$$

$$\sqrt[3]{3} = 3^{\frac{1}{3}} = 3^{\frac{2}{6}} = \sqrt[6]{3^2} = \sqrt[6]{9}.$$

We have, therefore, the following rule:

Write the radicals with fractional exponents, and change these fractional exponents to equivalent exponents having the least common denominator. Raise each radical to the power denoted by the numerator, and indicate the root denoted by the common denominator.

Exercise 98.

Reduce to surds of the same order:

1. $\sqrt[4]{3}$ and $\sqrt[6]{5}$.
2. $\sqrt[3]{14}$ and $\sqrt{6}$.
3. $\sqrt{2}$ and $\sqrt[3]{4}$.
4. $\sqrt{a}$ and $\sqrt[3]{b^2}$.
5. $\sqrt{5}$ and $\sqrt[6]{75}$.
6. $2^{\frac{1}{2}}$, $2^{\frac{3}{4}}$, and $2^{\frac{2}{3}}$.
7. $\sqrt{2}$, $\sqrt[3]{3}$, and $\sqrt[4]{5}$.
8. $\sqrt[6]{a^2}$, $\sqrt[3]{b}$, and $\sqrt{c}$.
9. $\sqrt[5]{a^4}$, $\sqrt[10]{c^3}$, and $\sqrt{x^3}$.
10. $\sqrt[4]{x^2y}$, $\sqrt[3]{abc}$, and $\sqrt[6]{2z}$.
11. $\sqrt[6]{x-y}$ and $\sqrt[4]{x+y}$.
12. $\sqrt[3]{a+b}$ and $\sqrt{a-b}$.

Note. Surds of different orders may be reduced to surds of the same order and then compared in respect to magnitude.

Arrange in order of magnitude:

13. $\sqrt[3]{15}$ and $\sqrt{6}$.
14. $\sqrt[3]{4}$ and $\sqrt{3}$.
15. $\sqrt[6]{80}$, $\sqrt[3]{9}$, and $\sqrt{8}$.
16. $\sqrt{3}$, $\sqrt[3]{5}$, and $\sqrt[4]{7}$.

Addition and Subtraction of Radicals.

262. In the addition of surds, each surd must be reduced to its simplest form; and, if the resulting surds are similar,

Find the algebraic sum of the coefficients, and to this sum annex the common surd factor.

If the resulting surds are not similar,

Connect them with their proper signs.

1. Simplify $\sqrt{27} + \sqrt{48} + \sqrt{147}$.

$$\sqrt{27} = (3^2 \times 3)^{\frac{1}{2}} = 3 \times 3^{\frac{1}{2}} = 3\sqrt{3};$$
$$\sqrt{48} = (2^4 \times 3)^{\frac{1}{2}} = 2^2 \times 3^{\frac{1}{2}} = 4 \times 3^{\frac{1}{2}} = 4\sqrt{3};$$
$$\sqrt{147} = (7^2 \times 3)^{\frac{1}{2}} = 7 \times 3^{\frac{1}{2}} = 7\sqrt{3}.$$
$$\sqrt{27} + \sqrt{48} + \sqrt{147} = (3 + 4 + 7)\sqrt{3} = 14\sqrt{3}.$$

2. Simplify $2\sqrt[3]{320} - 3\sqrt[3]{40}$.

$$2\sqrt[3]{320} = 2(2^6 \times 5)^{\frac{1}{3}} = 2 \times 2^2 \times 5^{\frac{1}{3}} = 8\sqrt[3]{5};$$
$$3\sqrt[3]{40} = 3(2^3 \times 5)^{\frac{1}{3}} = 3 \times 2 \times 5^{\frac{1}{3}} = 6\sqrt[3]{5}.$$
$$\therefore 2\sqrt[3]{320} - 3\sqrt[3]{40} = (8 - 6)\sqrt{5} = 2\sqrt[3]{5}.$$

3. Simplify $2\sqrt{\frac{5}{3}} - 3\sqrt{\frac{3}{5}} + \sqrt{\frac{4}{15}}$.

$$2\sqrt{\tfrac{5}{3}} = 2\sqrt{\tfrac{15}{9}} = 2\sqrt{15 \times \tfrac{1}{9}} = \tfrac{2}{3}\sqrt{15};$$
$$3\sqrt{\tfrac{3}{5}} = 3\sqrt{\tfrac{15}{25}} = 3\sqrt{15 \times \tfrac{1}{25}} = \tfrac{3}{5}\sqrt{15};$$
$$\sqrt{\tfrac{4}{15}} = \sqrt{\frac{4 \times 15}{15^2}} = \sqrt{15 \times \frac{4}{15^2}} = \tfrac{2}{15}\sqrt{15}.$$
$$\therefore 2\sqrt{\tfrac{5}{3}} - 3\sqrt{\tfrac{3}{5}} + \sqrt{\tfrac{4}{15}} = (\tfrac{2}{3} - \tfrac{3}{5} + \tfrac{2}{15})\sqrt{15} = \tfrac{1}{5}\sqrt{15}.$$

EXERCISE 99.

Simplify:

1. $4\sqrt{11} + 3\sqrt{11} - 5\sqrt{11}$.
2. $2\sqrt{3} - 5\sqrt{3} + 9\sqrt{3}$.
3. $5\sqrt[3]{4} + 2\sqrt[3]{32} - \sqrt[3]{108}$.
4. $3\sqrt[5]{2} + 4\sqrt[5]{2} - \sqrt[5]{64}$.
5. $\frac{1}{2}\sqrt[3]{5} + 2\frac{1}{2}\sqrt[3]{5} + \frac{1}{4}\sqrt[3]{40}$.
6. $3\sqrt[4]{3} - 5\sqrt[4]{48} + \sqrt[4]{243}$.
7. $\sqrt{27} + \sqrt{48} + \sqrt{75}$.
8. $4\sqrt{147} + 3\sqrt{75} + \sqrt{192}$.
9. $\sqrt{a} + \frac{1}{2}\sqrt{a} + \frac{3}{2}\sqrt{a}$.
10. $\sqrt[3]{a^2} + \frac{1}{2}\sqrt[3]{a^2} - 3\sqrt[3]{27a^2}$.

11. $\sqrt{a^3} + b\sqrt{a} - 3\sqrt{a}.$

12. $\sqrt{25\,b} + 2\sqrt{9\,b} - 3\sqrt{4\,b}.$

13. $2\sqrt{175} - 3\sqrt{63} + 5\sqrt{28}.$

14. $\sqrt{2} + 3\sqrt{32} + \frac{1}{2}\sqrt{128} - 6\sqrt{18}.$

15. $\sqrt{75} + \sqrt{48} - \sqrt{147} + \sqrt{300}.$

16. $20\sqrt{245} - \sqrt{5} + \sqrt{125} - 2\frac{1}{2}\sqrt{180}.$

17. $2\sqrt{20} + \frac{1}{2}\sqrt{12} - 2\sqrt{27} + 5\sqrt{45} - 9\sqrt{12}.$

18. $7\sqrt{25} + 4\sqrt{45} - \sqrt{9} - 2\sqrt{80} + \sqrt{20} - 4\sqrt{64}.$

19. $\sqrt[3]{54} + \sqrt{\frac{1}{2}} - \sqrt[3]{250} - \frac{3}{4}\sqrt{\frac{2}{9}}.$

20. $2\sqrt{\frac{5}{3}} + \sqrt{60} - \sqrt{15} + \sqrt{\frac{3}{5}} + \sqrt{\frac{4}{15}}.$

21. $\sqrt[3]{27\,c^4} - \sqrt[3]{8\,c^4} + \sqrt[3]{125\,c}.$

22. $\sqrt[5]{a^5b} - \sqrt[5]{b^6} + \sqrt[5]{32\,b}.$

23. $\sqrt{a^4x} + \sqrt{b^4x} - \sqrt{4\,a^2b^2x}.$

24. $\sqrt{4\,x^2y^2z} + \sqrt{y^4z} + \sqrt{x^4z}.$

25. $\sqrt{a^2b^2c} - a\sqrt{4\,c} + b\sqrt{a^2c}.$

26. $\sqrt[4]{81\,a^5} - \sqrt[4]{16\,a} + \sqrt[4]{256\,a^5}.$

27. $\sqrt[3]{27\,m^4} - \sqrt[3]{125\,m} + \sqrt[3]{216\,m}.$

28. $\sqrt{8\,a} - \sqrt{50\,a^3} - 3\sqrt{18\,a}.$

29. $6\,a\sqrt{63\,ab^3} - 3\sqrt{112\,a^3b^3} + 2\,ab\sqrt{343\,ab}.$

30. $3\sqrt{125\,m^3n^2} + n\sqrt{20\,m^3} - \sqrt{500\,m^3n^2}.$

31. $\sqrt{32\,a^4b^5} + 6\sqrt{72\,b} + 3\sqrt{128\,a^2b^3}.$

32. $2\sqrt[3]{a^6b} - 3\,a^2\sqrt[3]{64\,b} + 5\,a\sqrt[3]{a^3b} + 2\,a^2\sqrt[3]{125\,b}.$

Multiplication of Radicals.

263. Since $\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab}$, we have

1. $3\sqrt{8} \times 5\sqrt{2} = 3 \times 5 \times \sqrt{8} \times \sqrt{2} = 15\sqrt{16} = 60$;
2. $3\sqrt{2} \times 4\sqrt[3]{3} = 3\sqrt[6]{2^3} \times 4\sqrt[6]{3^2} = 12\sqrt[6]{72}$.

We have, therefore, the following rule:

Express the radicals with a common index. Find the product of the coefficients for the required coefficient, and the product of the surd factors for the required surd factor.

Reduce the result to its simplest form.

EXERCISE 100.

Find the product of:

1. $\sqrt{3} \times \sqrt{27}$.
2. $\sqrt{5} \times \sqrt{20}$.
3. $\sqrt{2} \times \sqrt{18}$.
4. $\sqrt[3]{3} \times \sqrt[3]{9}$.
5. $\sqrt[3]{2} \times \sqrt[3]{32}$.
6. $\sqrt[4]{27} \times \sqrt[4]{3}$.
7. $\sqrt[5]{4} \times \sqrt[5]{8}$.
8. $\sqrt[5]{27} \times \sqrt[5]{9}$.
9. $\sqrt{2} \times \sqrt{12}$.
10. $\sqrt{3} \times \sqrt{6}$.
11. $\sqrt[3]{3} \times \sqrt[3]{18}$.
12. $\sqrt[4]{6} \times \sqrt[4]{8}$.
13. $\sqrt[5]{54} \times \sqrt[5]{9}$.
14. $2\sqrt{8} \times \sqrt{2}$.
15. $\sqrt[5]{8} \times \sqrt[5]{-4}$.
16. $\sqrt[3]{7} \times \sqrt[3]{-49}$.
17. $\sqrt[3]{81} \times \sqrt[3]{-45}$.
18. $\frac{2}{3}\sqrt[3]{18} \times \frac{3}{4}\sqrt[3]{3}$.
19. $(\sqrt{18} + 2\sqrt{72} - 3\sqrt{8}) \times \sqrt{2}$.
20. $(\sqrt[3]{32} - \frac{1}{2}\sqrt[3]{864} + 3\sqrt[3]{4}) \times \sqrt[3]{2}$.
21. $(\frac{1}{3}\sqrt{27} - \frac{1}{4}\sqrt{2187} + \frac{1}{8}\sqrt{432}) \times \sqrt{3}$.
22. $\sqrt{5} \times \sqrt[3]{4}$.
23. $\sqrt[3]{16} \times \sqrt[6]{250}$.
24. $\sqrt[4]{64} \times \sqrt[8]{16}$.
25. $\sqrt{3} \times \sqrt[3]{72}$.
26. $\sqrt{\frac{2}{9}} \times \sqrt[3]{\frac{1}{2}}$.
27. $\sqrt{\frac{7}{12}} \times \sqrt[3]{\frac{3}{7}}$.
28. $\sqrt[3]{81} \times \sqrt{3}$.
29. $\sqrt[3]{\frac{5}{6}} \times \sqrt{\frac{3}{5}}$.
30. $\sqrt[4]{\frac{2}{3}} \times \sqrt[3]{\frac{1}{2}}$.

264. Compound radicals are multiplied as follows:

Multiply $2\sqrt{3}+3\sqrt{x}$ by $3\sqrt{3}-4\sqrt{x}$.

$$\begin{array}{l}
2\sqrt{3}+3\sqrt{x}\\
\underline{3\sqrt{3}-4\sqrt{x}}\\
18+9\sqrt{3x}\\
\underline{-8\sqrt{3x}-12x}\\
18+\sqrt{3x}-12x
\end{array}$$

EXERCISE 101.

Multiply:

1. $\sqrt{5+\sqrt{4}}$ by $\sqrt{5-\sqrt{4}}$.
2. $\sqrt{9-\sqrt{17}}$ by $\sqrt{9+\sqrt{17}}$.
3. $3+2\sqrt{5}$ by $2-\sqrt{5}$.
4. $8+3\sqrt{2}$ by $2-\sqrt{2}$.
5. $5+2\sqrt{3}$ by $3-5\sqrt{3}$.
6. $3-\sqrt{6}$ by $6-3\sqrt{6}$.
7. $2\sqrt{6}-3\sqrt{5}$ by $\sqrt{3}+2\sqrt{2}$.
8. $7-\sqrt{3}$ by $\sqrt{2}+\sqrt{5}$.
9. $\sqrt[3]{9}-2\sqrt[3]{4}$ by $4\sqrt[3]{3}+\sqrt[3]{2}$.
10. $2\sqrt{30}-3\sqrt{5}+5\sqrt{3}$ by $\sqrt{8}+\sqrt{3}-\sqrt{5}$.
11. $3\sqrt{5}-2\sqrt{3}+4\sqrt{7}$ by $3\sqrt{7}-4\sqrt{5}-5\sqrt{3}$.
12. $4\sqrt{8}+\frac{1}{2}\sqrt{12}-\frac{1}{4}\sqrt{32}$ by $8\sqrt{32}-4\sqrt{50}-2\sqrt{2}$.
13. $\sqrt[3]{6}-\sqrt[3]{3}+\sqrt[3]{16}$ by $\sqrt[3]{36}+\sqrt[3]{9}-\sqrt[3]{4}$.
14. $2\sqrt{\frac{2}{3}}-8\sqrt{\frac{3}{8}}+3\sqrt{\frac{3}{2}}$ by $3\sqrt{\frac{2}{3}}-\sqrt{12}-\sqrt{6}$.
15. $2\sqrt{\frac{5}{6}}-4\sqrt{\frac{3}{2}}-7\sqrt{\frac{6}{5}}$ by $3\sqrt{\frac{5}{6}}-5\sqrt{30}-2\sqrt{\frac{15}{2}}$.
16. $2\sqrt{12}+3\sqrt{3}+6\sqrt{\frac{1}{8}}$ by $2\sqrt{12}+3\sqrt{3}+6\sqrt{\frac{1}{}}$

Division of Radicals.

265. Since $\dfrac{\sqrt[n]{ab}}{\sqrt[n]{a}} = \dfrac{\sqrt[n]{a} \times \sqrt[n]{b}}{\sqrt[n]{a}} = \sqrt[n]{b}$, we have

1. $$\frac{4\sqrt{8}}{2\sqrt{2}} = 2\sqrt{4} = 4;$$

2. $$\frac{4\sqrt[3]{3}}{2\sqrt{2}} = \frac{4\sqrt[6]{3^2}}{2\sqrt[6]{2^3}} = \frac{4\sqrt[6]{3^2 \times 2^3}}{2\sqrt[6]{2^6}} = \sqrt[6]{72}.$$

We have, therefore, the following rule:

Express the radicals with a common index. Find the quotient of the coefficients for the required coefficient, and the quotient of the surd factors for the required surd factor.

Reduce the result to its simplest form.

Exercise 102.

Divide:

1. $\sqrt{243}$ by $\sqrt{3}$.
2. $\sqrt[3]{81}$ by $\sqrt[3]{3}$.
3. $\sqrt{3a^7}$ by $\sqrt{a^3}$.
4. $\sqrt{\frac{1}{2}}$ by $\sqrt{\frac{2}{9}}$.
5. $\sqrt{\frac{5}{6}}$ by $\sqrt{\frac{3}{5}}$.
6. $\sqrt{\frac{7}{12}}$ by $\sqrt{\frac{3}{7}}$.
7. $\sqrt{\frac{14}{15}}$ by $\sqrt{\frac{24}{35}}$.
8. $\sqrt{\frac{22}{27}}$ by $\sqrt{\frac{6}{11}}$.
9. $\sqrt{\frac{52}{85}}$ by $\sqrt{\frac{17}{65}}$.
10. $3\sqrt{6} + 45\sqrt{2}$ by $3\sqrt{3}$.
11. $42\sqrt{5} - 30\sqrt{3}$ by $2\sqrt{15}$.
12. $84\sqrt{15} + 168\sqrt{6}$ by $3\sqrt{21}$.
13. $30\sqrt[3]{4} - 36\sqrt[3]{10} + 30\sqrt[3]{90}$ by $3\sqrt[3]{20}$.
14. $50\sqrt[3]{18} + 18\sqrt[3]{20} - 48\sqrt[3]{5}$ by $2\sqrt[3]{30}$.
15. $\sqrt{54}$ by $\sqrt[4]{36}$.
16. $\sqrt[3]{49}$ by $\sqrt{7}$.
17. $\sqrt[3]{12}$ by $\sqrt{6}$.
18. $\sqrt{\frac{8}{45}}$ by $\sqrt[4]{6\frac{1}{4}}$.
19. $\sqrt{\frac{3}{5}}$ by $\sqrt[6]{3\frac{3}{8}}$.
20. $\sqrt[3]{2x}$ by $\sqrt{x^3}$.
21. $\sqrt[6]{0.064}$ by $\sqrt{10}$.
22. $\sqrt{x^2 - y^2}$ by $x + y$.

266. The quotient of one surd divided by another may be found by *rationalizing the divisor;* that is, by multiplying the dividend and divisor by a factor that will free the divisor from surds. This method is of great utility when we wish to find the approximate numerical value of the quotient of two simple surds, and is the method required when the divisor is a compound surd.

1. Divide $3\sqrt{8}$ by $\sqrt{6}$.

$$\frac{3\sqrt{8}}{\sqrt{6}} = \frac{6\sqrt{2}}{\sqrt{6}} = \frac{6\sqrt{2} \times \sqrt{6}}{\sqrt{6} \times \sqrt{6}} = \frac{6\sqrt{12}}{6} = \sqrt{12} = 2\sqrt{3}.$$

2. Divide $3\sqrt{5} - 4\sqrt{2}$ by $2\sqrt{5} + 3\sqrt{2}$.

Multiply the dividend and the divisor by $2\sqrt{5} - 3\sqrt{2}$,

$$\frac{(3\sqrt{5} - 4\sqrt{2})(2\sqrt{5} - 3\sqrt{2})}{(2\sqrt{5} + 3\sqrt{2})(2\sqrt{5} - 3\sqrt{2})} = \frac{54 - 17\sqrt{10}}{20 - 18}$$

$$= \frac{54 - 17\sqrt{10}}{2} = 27 - \tfrac{17}{2}\sqrt{10}.$$ Hence,

267. When the divisor is a binomial containing surds of the second order only,

Multiply the dividend and the divisor by the divisor, with the sign between the terms changed.

Exercise 103.

Divide:

1. $\sqrt{a} + \sqrt{b}$ by $\sqrt{ab}$.
2. $\sqrt{125}$ by $5\sqrt{65}$.
3. 3 by $11 + 3\sqrt{7}$.
4. $3\sqrt{2} - 1$ by $3\sqrt{2} + 1$.
5. 17 by $3\sqrt{7} + 2\sqrt{3}$.
6. 1 by $\sqrt{2} + \sqrt{3}$.
7. $3 + 5\sqrt{7}$ by $3 - 5\sqrt{7}$.
8. $21\sqrt{3}$ by $4\sqrt{3} - 3\sqrt{2}$.
9. $75\sqrt{14}$ by $8\sqrt{2} + 2\sqrt{7}$.
10. $\sqrt{5} - \sqrt{3}$ by $\sqrt{5} + \sqrt{3}$.
11. $\sqrt{8} + \sqrt{7}$ by $\sqrt{7} - \sqrt{2}$.
12. $7 - 3\sqrt{10}$ by $5 + 4\sqrt{5}$.

Given $\sqrt{2} = 1.41421$, $\sqrt{3} = 1.73205$, $\sqrt{5} = 2.23607$; find to four places of decimals the value of:

13. $\frac{10}{\sqrt{2}}$.

14. $\frac{8}{\sqrt{3}}$.

15. $\frac{12}{\sqrt{5}}$.

16. $\frac{1}{\sqrt{500}}$.

17. $\frac{1}{\sqrt{243}}$.

18. $\frac{1}{2\sqrt{3}}$.

19. $\frac{1}{3\sqrt{2}}$.

20. $\frac{1}{\sqrt{125}}$.

21. $\frac{1}{4\sqrt{5}}$.

22. $\frac{7-3\sqrt{5}}{5+4\sqrt{5}}$.

23. $\frac{3+\sqrt{5}}{\sqrt{5}-2}$.

24. $\frac{3\sqrt{2}-1}{3\sqrt{2}+1}$.

Involution and Evolution of Radicals.

268. Any power or root of a radical is easily found by using fractional exponents.

1. Find the square of $2\sqrt[3]{a}$.

$$(2\sqrt[3]{a})^2 = (2\,a^{\frac{1}{3}})^2 = 2^2\,a^{\frac{2}{3}} = 4\,a^{\frac{2}{3}} = 4\sqrt[3]{a^2}.$$

2. Find the cube of $2\sqrt{a}$.

$$(2\sqrt{a})^3 = (2\,a^{\frac{1}{2}})^3 = 2^3\,a^{\frac{3}{2}} = 8\,a^{\frac{3}{2}} = 8\,a\sqrt{a}.$$

3. Find the square root of $4\,x\sqrt{a^3b^3}$.

$$(4\,x\sqrt{a^3b^3})^{\frac{1}{2}} = (4\,xa^{\frac{3}{2}}b^{\frac{3}{2}})^{\frac{1}{2}} = 4^{\frac{1}{2}}\,x^{\frac{1}{2}}a^{\frac{3}{4}}b^{\frac{3}{4}} = 4^{\frac{1}{2}}\,x^{\frac{2}{4}}a^{\frac{3}{4}}b^{\frac{3}{4}} = 2\sqrt[4]{a^3b^3x^2}.$$

4. Find the cube root of $4\,x\sqrt{a^3b^3}$.

$$(4\,x\sqrt{a^3b^3})^{\frac{1}{3}} = (4\,xa^{\frac{3}{2}}b^{\frac{3}{2}})^{\frac{1}{3}} = 4^{\frac{1}{3}}\,x^{\frac{1}{3}}a^{\frac{1}{2}}b^{\frac{1}{2}} = 4^{\frac{2}{6}}\,x^{\frac{2}{6}}a^{\frac{3}{6}}b^{\frac{3}{6}} = \sqrt[6]{16\,a^3b^3x^2}.$$

Exercise 104.

Perform the operations indicated:

1. $(\sqrt[3]{m^2})^3$.
2. $(\sqrt{m^3})^5$.
3. $(\sqrt[5]{x^4})^{15}$.
4. $(\sqrt[3]{y^5})^{12}$.
5. $\sqrt[4]{\sqrt{(x-y)^8}}$.
6. $\sqrt[3]{\sqrt[6]{(a-b)^{36}}}$.

7. $(\sqrt{2a^3b})^4$.

8. $(\sqrt[3]{x^2 - y^2})^6$.

9. $(\sqrt[8]{x})^4$.

10. $\sqrt{\sqrt[3]{a^2}}$.

11. $\sqrt[4]{\sqrt[3]{729}}$.

12. $\sqrt[3]{\sqrt{125}}$.

13. $\sqrt[7]{\sqrt[3]{(3a - 2b)^{14}}}$.

14. $\sqrt[3]{\sqrt[5]{32a^{45}}}$.

15. $\sqrt[7]{128\sqrt[5]{243a^{70}}}$.

Properties of Quadratic Surds.

269. **A quadratic surd** is the indicated root of an imperfect square, as $\sqrt{2}$.

270. THEOREM 1. *The product or quotient of two dissimilar quadratic surds will be a quadratic surd.* Thus,

Thus,
$$\sqrt{ab} \times \sqrt{abc} = ab\sqrt{c};$$
$$\sqrt{abc} \div \sqrt{ab} = \sqrt{c}.$$

Two *dissimilar* quadratic surds cannot have *all* the factors under the radical sign *alike*. Hence, their product or quotient will contain the *first power only of at least one factor*, and will therefore be a surd.

271. THEOREM 2. *The sum or difference of two dissimilar quadratic surds cannot be a rational number, nor can the sum or difference be expressed as a single surd.*

For, if $\sqrt{a} \pm \sqrt{b}$ could equal a rational number c, we should have, by squaring,

$$a \pm 2\sqrt{ab} + b = c^2;$$

that is,
$$\pm 2\sqrt{ab} = c^2 - a - b.$$

Now, as the right side of this equation is rational, the left side would be rational; but, by § 270, $\sqrt{ab}$ cannot be rational. Therefore, $\sqrt{a} \pm \sqrt{b}$ cannot be rational.

In like manner it may be shown that $\sqrt{a} \pm \sqrt{b}$ cannot be expressed as a single surd $\sqrt{c}$.

272. THEOREM 3. *A quadratic surd cannot equal the sum of a rational number and a surd.*

For, if $\sqrt{a}$ could equal $c+\sqrt{b}$, we should have, by squaring,

$$a = c^2 + 2c\sqrt{b} + b,$$

and, by transposing,

$$2c\sqrt{b} = a - b - c^2.$$

That is, a surd equal to a rational number, which is impossible.

273. THEOREM 4. *If* $a+\sqrt{b} = x+\sqrt{y}$, *then* a *will equal* x, *and* b *will equal* y.

For, by transposing, $\sqrt{b}-\sqrt{y} = x - a$; and if b were not equal to y, the difference of two unequal surds would be rational, which by § 271 is impossible.

$$\therefore b = y, \text{ and } a = x.$$

In like manner, if $a-\sqrt{b} = x-\sqrt{y}$, a will equal x, and b will equal y.

274. THEOREM 5. *If* $\sqrt{a+\sqrt{b}} = \sqrt{x}+\sqrt{y}$, *then* $\sqrt{a-\sqrt{b}} = \sqrt{x}-\sqrt{y}$.

Square both sides of the given equation,

$$a+\sqrt{b} = x + 2\sqrt{xy} + y.$$

Therefore, by § 273,
$$a = x + y, \qquad (1)$$
and
$$\sqrt{b} = 2\sqrt{xy}. \qquad (2)$$

Subtract (2) from (1),

$$a-\sqrt{b} = x - 2\sqrt{xy} + y.$$

Extract the square root of both sides,

$$\sqrt{a-\sqrt{b}} = \sqrt{x}-\sqrt{y}.$$

275. To Extract the Square Root of a Binomial Surd.

1. Extract the square root of $7 + 4\sqrt{3}$.

Let $\sqrt{x} + \sqrt{y} = \sqrt{7 + 4\sqrt{3}}$. (1)

Then, by § 274, $\sqrt{x} - \sqrt{y} = \sqrt{7 - 4\sqrt{3}}$. (2)

Multiply (1) by (2), $x - y = \sqrt{49 - 48}$.

$\therefore x - y = 1$.

Square (1), then § 273, $x + y = 7$.

$\therefore x = 4$, and $y = 3$.

$\therefore \sqrt{x} + \sqrt{y} = \sqrt{4} + \sqrt{3}$.

$\therefore \sqrt{7 + 4\sqrt{3}} = 2 + \sqrt{3}$.

A root may be found by inspection, when the given expression can be written in the form $a + 2\sqrt{b}$, by finding two numbers that have their sum equal to a and their product equal to b.

2. Find by inspection the square root of $75 - 12\sqrt{21}$.

It is necessary that the coefficient of the surd be 2; therefore, $75 - 12\sqrt{21}$ must be put in the form $75 - 2\sqrt{6^2 \times 21}$;

that is, $75 - 2\sqrt{756}$.

Two numbers whose sum is 75 and product 756 are 63 and 12.

Then, $75 - 2\sqrt{756} = 63 - 2\sqrt{63 \times 12} + 12$

$= (\sqrt{63} - \sqrt{12})^2$.

That is, $\sqrt{63} - \sqrt{12} =$ the square root of $75 - 12\sqrt{21}$;

or $3\sqrt{7} - 2\sqrt{3} =$ the square root of $75 - 12\sqrt{21}$.

3. Extract the square root of $11 + 6\sqrt{2}$.

$11 + 6\sqrt{2} = 11 + 2\sqrt{18}$.

Two numbers whose sum is 11 and product 18 are 9 and 2.

Then, $11 + 2\sqrt{18} = 9 + 2\sqrt{9 \times 2} + 2$

$= (\sqrt{9} + \sqrt{2})^2$.

That is, $\sqrt{9} + \sqrt{2} =$ the square root of $11 + 6\sqrt{2}$,

or $3 + \sqrt{2} =$ the square root of $11 + 6\sqrt{2}$.

EXERCISE 105.

Find the square root of:

1. $7 - 4\sqrt{3}$.
2. $11 + \sqrt{72}$.
3. $7 + 2\sqrt{10}$.
4. $18 + 8\sqrt{5}$.
5. $8 + 2\sqrt{15}$.
6. $15 - 4\sqrt{14}$.
7. $16 + 5\sqrt{7}$.
8. $75 + 12\sqrt{21}$.
9. $19 + 8\sqrt{3}$.
10. $8\sqrt{6} + 20$.
11. $28 - 16\sqrt{3}$.
12. $51 + 36\sqrt{2}$.
13. $94 + 42\sqrt{5}$.
14. $11 - 2\sqrt{30}$.
15. $47 - 4\sqrt{33}$.
16. $29 + 6\sqrt{22}$.
17. $83 + 12\sqrt{35}$.
18. $55 - 12\sqrt{21}$.

Equations Containing Radicals.

276. An equation containing a *single* radical may be solved by arranging the terms so as to have the radical alone on one side, and then raising both sides to a power corresponding to the order of the radical.

Solve $\sqrt{x^2 - 9} + x = 9$.

Transpose x, $\qquad \sqrt{x^2 - 9} = 9 - x.$

Square, $\qquad x^2 - 9 = 81 - 18x + x^2.$

$$18x = 90.$$

$$\therefore x = 5.$$

277. If *two* radicals are involved, two steps may be necessary.

Solve $\sqrt{x + 15} + \sqrt{x} = 15$.

Square, $\qquad x + 15 + 2\sqrt{x^2 + 15x} + x = 225.$

Transpose, $\qquad 2\sqrt{x^2 + 15x} = 210 - 2x.$

Divide by 2, $\qquad \sqrt{x^2 + 15x} = 105 - x.$

Square, $\qquad x^2 + 15x = 11025 - 210x + x^2.$

$$225x = 11025.$$

$$\therefore x = 49.$$

EXERCISE 106.

Solve:

1. $2\sqrt{x+5}=\sqrt{28}.$

2. $3\sqrt{4x-8}=\sqrt{13x-3}.$

3. $\sqrt{x+9}=5\sqrt{x-3}.$

4. $4=2\sqrt{x}-3.$

5. $5-\sqrt{3y}=4.$

6. $7+2\sqrt[3]{3x}=5.$

7. $\sqrt[3]{2x-3}=-3.$

8. $\sqrt[3]{3x+7}=3.$

9. $14+\sqrt[3]{4x-40}=10.$

10. $\sqrt[3]{10y-4}=\sqrt[3]{7y+11}.$

11. $2\sqrt{x-2}=\sqrt[4]{32(x-2)^3}.$

12. $\sqrt{\frac{15}{4}+x}=\frac{3}{2}+\sqrt{x}.$

13. $\sqrt{32+x}=16-\sqrt{x}.$

14. $\sqrt{x}-\sqrt{x-5}=\sqrt{5}.$

15. $\sqrt{x+20}-\sqrt{x-1}-3=0.$

16. $\sqrt{x+15}-7=7-\sqrt{x-13}.$

17. $x=7-\sqrt{x^2-7}.$

18. $\sqrt{x-7}=\sqrt{x+1}-2.$

19. $\dfrac{\sqrt{x}-3}{\sqrt{x}+3}=\dfrac{\sqrt{x}+1}{\sqrt{x}-2}.$

20. $\dfrac{1}{2}-\dfrac{3}{x}=\sqrt{\dfrac{1}{4}-\dfrac{1}{x}\sqrt{9-\dfrac{36}{x}}}.$

21. $\dfrac{1+(1-x)^{\frac{1}{2}}}{1-(1-x)^{\frac{1}{2}}}=3.$

22. $(x-3)^{\frac{1}{2}}+x^{\frac{1}{2}}=\dfrac{3}{(x-3)^{\frac{1}{2}}}.$

23. $\sqrt{a+\sqrt{x}}+\sqrt{a-\sqrt{x}}=\sqrt{x}.$

24. $\sqrt{ax}-1=4+\frac{1}{2}\sqrt{ax}-\frac{1}{2}.$

25. $3\sqrt{x}-3\sqrt{a}=\sqrt{x}-\sqrt{a}+2\sqrt{a}.$

26. $\sqrt{9+2x}-\sqrt{2x}=\dfrac{5}{\sqrt{9+2x}}.$

CHAPTER XVIII.

IMAGINARY EXPRESSIONS

278. An **imaginary expression** is any expression which involves the *indicated even root of a negative number.*

It will be shown hereafter that *any* indicated even root of a negative number may be made to assume a form which involves only an indicated *square root* of a negative number. In considering imaginary expressions we accordingly need consider only expressions which involve the indicated square roots of negative numbers.

Imaginary expressions are also called **imaginary numbers** and **complex numbers.** In distinction from imaginary numbers all other numbers are called **real numbers.**

279. Imaginary Square Roots. If a and b are both positive, we have

$$\sqrt{ab} = \sqrt{a} \times \sqrt{b}.$$

If one of the two numbers a and b is positive and the other negative, it is *assumed* that the law still holds true; we have, accordingly:

$$\sqrt{-4} = \sqrt{4(-1)} = \sqrt{4} \times \sqrt{-1} = 2\sqrt{-1};$$
$$\sqrt{-5} = \sqrt{5(-1)} = \sqrt{5} \times \sqrt{-1} = 5^{\frac{1}{2}}\sqrt{-1};$$
$$\sqrt{-a} = \sqrt{a(-1)} = \sqrt{a} \times \sqrt{-1} = a^{\frac{1}{2}}\sqrt{-1};$$

and so on.

It appears, then, that every imaginary square root can be made to assume the form $a\sqrt{-1}$, where a is a real number.

280. The symbol $\sqrt{-1}$ is called the **imaginary unit,** and may be defined as an expression the square of which is -1.

Hence,
$$\begin{aligned}\sqrt{-1}\times\sqrt{-1} &= (\sqrt{-1})^2 = -1;\\ \sqrt{-a}\times\sqrt{-b} &= \sqrt{a}\times\sqrt{-1}\times\sqrt{b}\times\sqrt{-1}\\ &= \sqrt{a}\times\sqrt{b}\times(\sqrt{-1})^2\\ &= \sqrt{ab}\times(-1)\\ &= -\sqrt{ab}.\end{aligned}$$

281. It will be useful to form the successive powers of the imaginary unit.

$$\begin{aligned}&(\sqrt{-1}) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots &&= +\sqrt{-1};\\ &(\sqrt{-1})^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots &&= -1;\\ &(\sqrt{-1})^3 = (\sqrt{-1})^2\sqrt{-1} &&= (-1)\sqrt{-1} = -\sqrt{-1};\\ &(\sqrt{-1})^4 = (\sqrt{-1})^2(\sqrt{-1})^2 &&= (-1)(-1) = +1;\\ &(\sqrt{-1})^5 = (\sqrt{-1})^4\sqrt{-1} &&= (+1)\sqrt{-1} = +\sqrt{-1};\end{aligned}$$

and so on. If, therefore, n is zero or a positive integer,

$$\begin{aligned}(\sqrt{-1})^{4n+1} &= +\sqrt{-1};\\ (\sqrt{-1})^{4n+2} &= -1;\\ (\sqrt{-1})^{4n+3} &= -\sqrt{-1};\\ (\sqrt{-1})^{4n+4} &= +1.\end{aligned}$$

282. Every imaginary expression may be made to assume the form $a+b\sqrt{-1}$, where a and b are real numbers, and may be integers, fractions, or surds.

If $b=0$, the expression consists of only the real part a, and is therefore real.

If $a=0$, the expression consists of only the imaginary part $b\sqrt{-1}$, and is called a **pure imaginary.**

283. The form $a + b\sqrt{-1}$ is the **typical form** of imaginary expressions.

Reduce to the typical form $6 + \sqrt{-8}$.

This may be written $6 + \sqrt{8} \times \sqrt{-1}$, or $6 + 2\sqrt{2} \times \sqrt{-1}$; here $a = 6$, and $b = 2\sqrt{2}$.

284. Two expressions of the form $a + b\sqrt{-1}$, $a - b\sqrt{-1}$, are called **conjugate imaginaries.**

To find the sum and product of two conjugate imaginaries:

$$\begin{array}{ll} & a + b\sqrt{-1} \\ & a - b\sqrt{-1} \\ \hline \text{The sum is} & 2a \end{array}$$

$$\begin{array}{llll} & a & + b\sqrt{-1} & \\ & a & - b\sqrt{-1} & \\ \hline & a^2 & + ab\sqrt{-1} & \\ & & - ab\sqrt{-1} & + b^2 \\ \hline \text{The product is} & a^2 & & + b^2 \end{array}$$

From the above it appears that the *sum* and *product* of two conjugate imaginaries are both *real.*

285. THEOREM 1. *An imaginary expression cannot be equal to a real number.*

For, if possible, let

$$a + b\sqrt{-1} = c.$$

Transpose a, $\quad b\sqrt{-1} = c - a.$

Square, $\quad -b^2 = (c - a)^2.$

Since b^2 and $(c - a)^2$ are both positive, we have a negative number equal to a positive number, which is impossible.

286. Theorem 2. *If two imaginary expressions are equal, the real parts are equal and the imaginary parts are equal.*

For, let $a + b\sqrt{-1} = c + d\sqrt{-1}$.

Then, $(b - d)\sqrt{-1} = c - a$;

Square, $-(b - d)^2 = (c - a)^2$,

which is impossible unless $b = d$ and $a = c$.

287. Theorem 3. *If x and y are real and $x + y\sqrt{-1} = 0$, then $x = 0$ and $y = 0$.*

For, $y\sqrt{-1} = -x$.

Square, $-y^2 = x^2$.

Transpose $-y^2$, $x^2 + y^2 = 0$,

which is true only when $x = 0$ and $y = 0$.

Operations with Imaginaries.

1. Add $5 + 7\sqrt{-1}$ and $8 - 9\sqrt{-1}$.

The sum is $5 + 8 + 7\sqrt{-1} - 9\sqrt{-1}$,

or $13 - 2\sqrt{-1}$.

2. Multiply $3 + 2\sqrt{-1}$ by $5 - 4\sqrt{-1}$.

$$\begin{aligned}&(3 + 2\sqrt{-1})(5 - 4\sqrt{-1})\\ &= 15 - 12\sqrt{-1} + 10\sqrt{-1} - 8(-1)\\ &= 23 - 2\sqrt{-1}.\end{aligned}$$

3. Divide $14 + 5\sqrt{-1}$ by $2 - 3\sqrt{-1}$.

$$\begin{aligned}\frac{14 + 5\sqrt{-1}}{2 - 3\sqrt{-1}} &= \frac{(14 + 5\sqrt{-1})(2 + 3\sqrt{-1})}{(2 - 3\sqrt{-1})(2 + 3\sqrt{-1})}\\ &= \frac{13 + 52\sqrt{-1}}{4 - (-9)}\\ &= \frac{13 + 52\sqrt{-1}}{13}\\ &= 1 + 4\sqrt{-1}.\end{aligned}$$

EXERCISE 107.

Reduce to the form $b\sqrt{-1}$:

1. $\sqrt{-9}$.
2. $\sqrt{-16}$.
3. $\sqrt{-25}$.
4. $\sqrt{-144}$.
5. $\sqrt{-169}$.
6. $\sqrt{-x^2}$.
7. $\sqrt{-81}$.
8. $\sqrt{-256}$.
9. $\sqrt{-625}$.
10. $\sqrt{-36}$.
11. $\sqrt[6]{-64}$.
12. $\sqrt[6]{-729}$.
13. $\sqrt{-289}$.
14. $\sqrt[10]{-1024}$.
15. $\sqrt{-x^8}$.
16. $\sqrt{-x^9}$.
17. $\sqrt[6]{-x^{18}}$.
18. $\sqrt{-\frac{1}{4}}$.
19. $\sqrt{-a^4b^{-2}}$.
20. $\sqrt{-9x^4}$.
21. $\sqrt[6]{-(2x-3y)^{12}}$.
22. $\sqrt[10]{-(x-2y)^{20}}$.
23. $\sqrt{-(x^2+y^2)}$.
24. $\sqrt{-(x^2-y^2)}$.

Add:

25. $\sqrt{-25}+\sqrt{-49}-\sqrt{-121}$.
26. $\sqrt{-64}+\sqrt{-1}-\sqrt{-36}$.
27. $\sqrt{-a^4}+\sqrt{-4a^4}+\sqrt{-16a^4}$.
28. $\sqrt{-a^2}+\sqrt{-81a^2}-\sqrt{-a^2}$.
29. $a-b\sqrt{-1}+a+b\sqrt{-1}$.
30. $2+3\sqrt{-1}-2+3\sqrt{-1}$.
31. $a+b\sqrt{-1}+c-d\sqrt{-1}$.
32. $3a\sqrt{-1}-(2a-b)\sqrt{-1}$.

Multiply:

33. $\sqrt{-3}$ by $\sqrt{-5}$.
34. $-\sqrt{-5}$ by $\sqrt{-5}$.
35. $\sqrt{-16}$ by $\sqrt{-9}$.
36. $\sqrt{-x^2}$ by $\sqrt{-x}$.
37. $\sqrt{-x^2}$ by $\sqrt{-y^2}$.
38. $\sqrt{-8}$ by $\sqrt{-16}$.

39. $\sqrt{-25}$ by $\sqrt{-64}$.

40. $\sqrt{-(a+b)}$ by $\sqrt{-(a-b)}$.

41. $3\sqrt{-3}$ by $2\sqrt{-2}$.

42. $-5\sqrt{-2}$ by $2\sqrt{-5}$.

43. $\sqrt{-2}+\sqrt{-3}$ by $\sqrt{-4}-\sqrt{-5}$.

44. $x-\dfrac{1+\sqrt{-3}}{2}$ by $x-\dfrac{1-\sqrt{-3}}{2}$.

45. $a\sqrt{-a}+b\sqrt{-b}$ by $a\sqrt{-a}-b\sqrt{-b}$.

46. $2\sqrt{-2}+3\sqrt{-3}$ by $3\sqrt{-4}-2\sqrt{-5}$.

47. $\sqrt{3}+2\sqrt{-3}$ by $\sqrt{3}-2\sqrt{-3}$.

48. $m-3\sqrt{-b}$ by $n+4\sqrt{-c}$.

Perform the divisions indicated:

49. $\dfrac{a}{\sqrt{-1}}$.

50. $\dfrac{b}{\sqrt{-b^2}}$.

51. $\dfrac{c}{\sqrt{-4}}$.

52. $\dfrac{\sqrt{-9}}{\sqrt{-81}}$.

53. $\dfrac{\sqrt{-a}}{\sqrt{-b}}$.

54. $\dfrac{\sqrt{-ax}}{\sqrt{-x}}$.

55. $\dfrac{x}{\sqrt{-x}}$.

56. $\dfrac{x}{\sqrt{-x^2}}$.

57. $\dfrac{\sqrt{-x^2}}{-\sqrt{-x}}$.

58. $\dfrac{\sqrt{-8}}{\sqrt{-2}}$.

59. $\dfrac{\sqrt{-10x^3}}{\sqrt{-5x}}$.

60. $\dfrac{8\sqrt{-x^2}}{2\sqrt{x}}$.

61. $\dfrac{1}{3-\sqrt{-2}}$.

62. $\dfrac{2+\sqrt{-2}}{1-\sqrt{-1}}$.

63. $\dfrac{a+x\sqrt{-1}}{a-x\sqrt{-1}}$.

64. $\dfrac{\sqrt{5}+\sqrt{-6}}{\sqrt{6}-\sqrt{-8}}$.

65. $\dfrac{2a+3b\sqrt{-1}}{2a-3b\sqrt{-1}}$.

66. $\dfrac{\frac{1}{2}a-4b\sqrt{-1}}{4a-\frac{1}{2}b\sqrt{-1}}$

CHAPTER XIX.

QUADRATIC EQUATIONS.

288. We have already considered equations of the first degree in one or more unknowns. We pass now to the treatment of equations containing one or more unknowns to a degree not exceeding the second. An equation which contains the **square** of the unknown, but no higher power, is called a **quadratic equation.**

289 A quadratic equation which involves but one unknown number can contain only:

1. Terms involving the square of the unknown number.

2. Terms involving the first power of the unknown number.

3. Terms which do not involve the unknown number.

Collecting similar terms, every quadratic equation can be made to assume the form

$$ax^2 + bx + c = 0,$$

where a, b, and c are known numbers, and x the unknown number.

If a, b, c are numbers expressed by figures, the equation is a **numerical quadratic**. If a, b, c are numbers represented wholly or in part by letters, the equation is a **literal quadratic.**

In the equation $ax^2 + bx + c = 0$, a, b, and c are called the **coefficients** of the equation. The third term c is called the **constant** term.

290. If the first power of x is wanting, the equation is a **pure quadratic**; in this case $b = 0$.

If the first power of x is present, the equation is an **affected** or **complete quadratic.**

Pure Quadratic Equations.

1. Solve the equation $5x^2 - 48 = 2x^2$.

Collect the terms, $3x^2 = 48.$

Divide by 3, $x^2 = 16.$

Extract the square root, $x = \pm 4.$

It will be observed that there are *two* roots, and that these are numerically equal, but of opposite signs. There can be only two roots, since any number has only two square roots.

It may seem as though we ought to write the sign ± before the x as well as before the 4. If we do this, we have $+x = +4$, $-x = -4$, $+x = -4$, $-x = +4$.

From the first and second equations, $x = 4$; from the third and fourth, $x = -4$; these values of x are both given by the equation $x = \pm 4$. Hence it is *unnecessary* to write the ± sign on *both* sides of the reduced equation.

2. Solve the equation $3x^2 - 15 = 0$.

Transpose, $3x^2 = 15.$

Divide by 3, $x^2 = 5.$

Extract the square root, $x = \pm\sqrt{5}.$

The roots cannot be found exactly, since the square root of 5 cannot be found exactly; it can, however, be determined approximately to any required degree of accuracy; for example, the roots lie between 2.23606 and 2.23607; and between -2.23606 and -2.23607.

3. Solve the equation $3x^2 + 15 = 0$.

Transpose, $3x^2 = -15.$

Divide by 3, $x^2 = -5.$

Extract the square root, $x = \pm\sqrt{-5}.$

There is no square root of a negative number, since the square of any number, positive or negative, is necessarily positive.

The square root of -5 differs from the square root of $+5$ in that the latter can be found as accurately as we please, while the former cannot be found at all.

291. A root that can be found exactly is called an **exact** or **rational** root. Such roots are either whole numbers or fractions.

A root that is indicated but can be found only approximately is called a **surd.** Such roots involve the roots of imperfect powers.

Rational and surd roots are together called **real** roots.

A root that is indicated but cannot be found, either exactly or approximately, is called an **imaginary** root. Such roots involve the even roots of negative numbers.

EXERCISE 108.

Solve:

1. $3x^2 - 2 = x^2 + 6.$
2. $5x^2 + 10 = 6x^2 + 1.$
3. $7x^2 - 50 = 4x^2 + 25.$
4. $6x^2 - \frac{1}{6} = 4x^2 + \frac{11}{9}.$
5. $\frac{x^2+1}{5} = 10.$
6. $\frac{3x^2-8}{10} = 4.$
7. $\frac{x^2-9}{4} = \frac{x^2+1}{5}.$
8. $\frac{2x^2-4}{7} + \frac{x^2+4}{5} = 8.$
9. $\frac{3-x^2}{11} + \frac{x^2+5}{6} = 3.$
10. $\frac{5x^2+3}{8} - \frac{17-x^2}{4} = 4.$
11. $\frac{3}{4x^2} - \frac{1}{6x^2} = \frac{7}{3}.$
12. $\frac{5}{3x^2} - \frac{3}{5x^2} = \frac{4}{15}.$
13. $\frac{1}{x-1} - \frac{1}{x+1} = \frac{1}{4}.$
14. $\frac{15}{8-x} + \frac{7}{2-3x} = 2.$

15. $3x^2 + 11x = 10x + 8 + x^2 + x.$

16. $(x + 4)(x + 5) = 3(x + 1)(x + 2) - 4.$

17. $3(x - 2)(x + 3) = (x + 1)(x + 2) + x^2 + 5.$

18. $(2x + 1)(3x - 2) + (1 - x)(3 + 4x) = 3x^2 - 15.$

19. $\dfrac{x^2 + 9}{17} - \dfrac{2x^2 - 5}{9} + \dfrac{3x^2 + 10}{5} = 14.$

20. $\dfrac{3x^2 - 5}{7} + \dfrac{2x^2 + 4}{9} - \dfrac{x^2 - 3}{2} = 5.$

21. $\dfrac{10x^2 + 7}{18} - \dfrac{12x^2 + 2}{11x^2 - 8} = \dfrac{5x^2 - 9}{9}.$

22. $\dfrac{x - 1}{x + 1} + \dfrac{x + 1}{x - 1} = \dfrac{5}{2}.$

23. $ax^2 + b = c.$

24. $ax^2 + b = bx^2 + a.$

25. $x^2 + 2bx + c = b(2x + 1).$

26. $\dfrac{a}{x} + \dfrac{x}{a} = \dfrac{9a^2 - x^2}{ax}.$

27. $\dfrac{x + a}{x - a} + \dfrac{x - a}{x + a} = \dfrac{5}{2}.$

28. $\dfrac{2b}{x - b} + \dfrac{5x + 2b}{3x} = -1.$

29. $2\{(x + a)(x + b) + (x - a)(x - b)\} = a^2 + 4b^2.$

30. $2\{(x - a)(x + b) + (x + a)(x - b)\} = 9a^2 + 2ab + b^2.$

Affected Quadratic Equations.

292. Since $(x \pm b)^2 = x^2 \pm 2bx + b^2$, it is evident that the expression $x^2 \pm 2bx$ lacks only the *third term*, b^2, of being a perfect square.

This third term is the square of half the coefficient of x.

Every affected quadratic may be made to assume the form $x^2 \pm 2bx = c$, by dividing the equation through by the coefficient of x^2.

293. **To solve** such an equation:

The first step is to add to both members *the square of half the coefficient of* x. This is called completing the square.

The second step is to *extract the square root* of each member of the resulting equation.

The third step is to *reduce* the two resulting simple equations.

1. Solve the equation $x^2 - 8x = 20$.

Complete the square, $x^2 - 8x + 16 = 36$.

Extract the square root, $x - 4 = \pm 6$.

Reduce, $x = 4 + 6 = 10$,

or $x = 4 - 6 = -2$.

The roots are 10 and -2.

Verify by putting these numbers for x in the given equation.

$$\begin{array}{r|r} x = 10, & x = -2, \\ 10^2 - 8(10) = 20, & (-2)^2 - 8(-2) = 20, \\ 100 - 80 = 20. & 4 + 16 = 20. \end{array}$$

2. Solve the equation $\dfrac{x+1}{x-1} = \dfrac{4x-3}{x+9}$.

Free from fractions, $(x+1)(x+9) = (x-1)(4x-3)$.

Simplify, $-3x^2 + 17x = -6$.

Divide by -3, $x^2 - \frac{17}{3}x = 2$.

Half the coefficient of x is $\frac{1}{2}$ of $-\frac{17}{3} = -\frac{17}{6}$, and the square of $-\frac{17}{6}$ is $\frac{289}{36}$. Add the square of $-\frac{17}{6}$ to both sides, and we have

$$x^2 - \frac{17x}{3} + \left(\frac{17}{6}\right)^2 = 2 + \frac{289}{36} = \frac{361}{36}.$$

Extract the root, $x - \dfrac{17}{6} = \pm \dfrac{19}{6}$.

Transpose $-\frac{17}{6}$, $x = \dfrac{17 \pm 19}{6}$.

$$\therefore x = 6, \text{ or } -\frac{1}{3}.$$

EXERCISE 109.

Solve:

1. $x^2 + 2x = 8.$
2. $x^2 - 6x = 7.$
3. $x^2 - 4x = 12.$
4. $x^2 + 4x = 5.$
5. $x^2 + 5x = 14.$
6. $x^2 - 3x = 28.$
7. $2x^2 + x = 15.$
8. $5x^2 + 3x = 2.$
9. $x^2 + \frac{2}{3}x = 40.$
10. $3x^2 - 4x = 4.$
11. $6x^2 + x = 1.$
12. $6x^2 - x = 2.$
13. $12x^2 - 11x + 2 = 0.$
14. $15x^2 - 2x - 1 = 0.$
15. $\frac{(x+1)(x+2)}{5} - \frac{(x-1)(x-2)}{2} = 3.$
16. $\frac{(2x-3)x}{4} - \frac{(x+4)(x-1)}{6} = 1.$
17. $\frac{3x+5}{x+4} + \frac{2x-5}{x-2} = 3.$
18. $\frac{x-6}{x-2} + \frac{x+5}{2x+1} = 1.$
19. $\frac{4-3x}{2+x} - \frac{1+2x}{1-x} = \frac{9}{2}.$
20. $\frac{x}{x+1} + \frac{x+1}{x} = \frac{13}{6}.$
21. $5x^2 - 4x = 1.$
22. $\frac{x+2}{x-1} - \frac{4-x}{2x} = \frac{7}{3}.$
23. $\frac{x+3}{x-2} = \frac{5x+8}{x+4}.$
24. $\frac{2x-1}{3} + \frac{3}{2x-1} = 2.$
25. $\frac{x+1}{x+2} + \frac{x+2}{x+1} = \frac{13}{6}.$
26. $7x^2 - 8x = -1.$

294. If the coefficient of x^2 is 4, 9, 16, or any other perfect square, we may complete the square by adding to each side *the square of the quotient obtained from dividing the second term by twice the square root of the first term.*

Solve $4x^2 - 23x = -30$.

The square root of $4x^2$ is $2x$, and $23x$ divided by twice $2x$ is $\frac{23}{4}$. Add the square of $\frac{23}{4}$ to both sides.

Then, $4x^2 - 23x + \left(\frac{23}{4}\right)^2 = \frac{529}{16} - 30 = \frac{49}{16}.$

Extract the root, $2x - \frac{23}{4} = \pm\frac{7}{4}$

Transpose, $2x = \frac{23 \pm 7}{4} = \frac{30}{4}$, or $\frac{16}{4}$.

$\therefore x = 3\frac{3}{4}$, or 2.

If the coefficient of x^2 is not a perfect square, we may multiply the equation by a number that will make the coefficient of x^2 a perfect square.

Solve $-3x^2 + 5x = -2$.

Since the *even* root of a *negative* number is impossible, it is necessary to change the sign of each term. The resulting equation is

$$3x^2 - 5x = 2.$$

Multiply by 3, $9x^2 - 15x = 6.$

Complete the square, $9x^2 - 15x + \frac{25}{4} = \frac{49}{4}.$

Extract the square root, $3x - \frac{5}{2} = \pm\frac{7}{2}.$

Reduce, $3x = \frac{5 \pm 7}{2}.$

$$3x = 6, \text{ or } -1.$$

$$\therefore x = 2, \text{ or } -\frac{1}{3}.$$

EXERCISE 110.

Solve:

1. $3x^2 - 2x = 8.$

2. $5x^2 - 6x = 27.$

3. $2x^2 + 3x = 5.$

4. $2x^2 - 5x = 7.$

5. $3x^2 + 7x = 6.$

6. $5x^2 - 7x = 24.$

7. $8x^2 + 3x = 26.$

8. $7x^2 + 5x = 150.$

9. $6x^2 + 5x = 14.$

10. $7x^2 - 2x = \frac{3}{4}.$

11. $8x^2 + 7x = 51.$

12. $7x^2 - 20x = 75.$

13. $11x^2 - 10x = 24.$

14. $3x^2 + \dfrac{2x}{3} = 25.$

15. $x^2 - \dfrac{3x}{4} = 3x + 1.$

16. $\dfrac{x^2}{2} - \dfrac{x}{3} = 2(x - 2).$

17. $\dfrac{2x^2}{3} + \dfrac{3x}{2} = 15.$

18. $\dfrac{3x}{4} - 2x^2 = \dfrac{1}{16}.$

19. $\frac{3}{5}x^2 + \frac{5}{3}x = \dfrac{20}{3}.$

20. $2x - 3 = \dfrac{2}{x}.$

21. $\dfrac{7x}{5} - \dfrac{5}{3x} = \dfrac{20}{3}.$

22. $\dfrac{2x}{3} + \dfrac{3}{2x} = \dfrac{10}{3}.$

23. $(x + 2)(2x + 1) + (x - 1)(3x + 2) = 57.$

24. $3x(2x + 5) - (x + 3)(3x - 1) = 1.$

25. $\dfrac{(2x + 5)(x - 3)}{3} + \dfrac{x(3x + 4)}{5} = 5.$

26. $\frac{2}{3}(5x^2 - 8x - 6) - \frac{1}{2}(x^2 - 3) = 2x + 1.$

27. $\dfrac{2}{x + 3} + \dfrac{5}{x} = 2.$

28. $\dfrac{5}{x - 1} - \dfrac{3}{2x + 1} = \dfrac{4}{3}.$

29. $\dfrac{2}{x - 1} = \dfrac{3}{x - 2} + \dfrac{2}{x - 4}.$

30. $\dfrac{x + 2}{x - 4} + \dfrac{x + 3}{x - 2} = -5.$

Another Method of Completing the Square.

295. If a complete quadratic is multiplied by *four times the coefficient of* x^2, fractions will be avoided.

Solve $3x^2 - 5x = 2$.

Multiply by 12, $$36x^2 - 60x = 24.$$
Complete the square, $$36x^2 - 60x + 25 = 49.$$
Extract the square root, $$6x - 5 = \pm 7.$$
Reduce, $$6x = 5 \pm 7.$$
$$6x = 12, \text{ or } -2.$$
$$\therefore x = 2, \text{ or } -\frac{1}{3}.$$

The number added to complete the square by this last method is *the square of the coefficient of* x in the original equation $3x^2 - 5x = 2$.

296. If the coefficient of x is an *even* number, we may multiply by the *coefficient of* x^2, and add to each member the square of *half the coefficient of* x in the given equation.

Solve $3x^2 + 4x = 20$.

Multiply by the coefficient of x^2 and add to each side the square of *half* the coefficient of x,

$$9x^2 + (\quad) + 4 = 64.$$
Extract the square root, $$3x + 2 = \pm 8.$$
Reduce, $$3x = -2 \pm 8.$$
$$3x = 6, \text{ or } -10.$$
$$\therefore x = 2, \text{ or } -3\tfrac{1}{3}.$$

NOTE. If a trinomial is a perfect square, its root is found by taking the square root of the *first* and *third* terms and connecting these roots by the *sign* of the middle term. It is not necessary, therefore, in completing the square, to write the middle term, but its place may be indicated by a parenthesis, as in this example.

Verify by putting the values of x in the given equation.

$x = 2.$	$x = -3\tfrac{1}{3}.$
$3(2)^2 + 4(2) = 20.$	$3(-3\tfrac{1}{3})^2 + 4(-3\tfrac{1}{3}) = 20.$
$12 + 8 = 20.$	$33\tfrac{1}{3} - 13\tfrac{1}{3} = 20.$

EXERCISE 111.

Solve:

1. $x^2 - \frac{2x}{3} + \frac{1}{12} = 0.$

2. $\frac{x^2}{2} - \frac{x}{3} = 2(x+2).$

3. $\frac{3x}{4} + \frac{4}{3x} = \frac{13}{6}.$

4. $\frac{x+1}{x+4} = \frac{2x-1}{x+6}.$

5. $\frac{x}{x+1} - \frac{x+3}{2(x+4)} = -\frac{1}{18}$

6. $\frac{2}{x-1} = \frac{3}{x-2} + \frac{2}{x-4}.$

7. $\frac{3x}{2(x+1)} - \frac{5}{8} = \frac{3x^2}{x^2-1} - \frac{23}{4(x-1)}.$

8. $\frac{11-3x}{1-x} + \frac{2(7-4x)}{1-2x} = 1.$

9. $\frac{x+1}{x^2-4} + \frac{1-x}{x+2} = \frac{2}{5(x-2)}.$

10. $\frac{2x+7}{2x-3} + \frac{3x-2}{x+1} = 5.$

11. $\frac{2x+3}{2(2x-1)} - \frac{7-x}{2(x+1)} = \frac{7-3x}{4-3x}.$

12. $\frac{2x-1}{3} - \frac{3}{x-8} = \frac{x-2}{x-8} + 5.$

13. $\frac{3x+2}{2x-1} + \frac{7-x}{2x+1} = \frac{7x-1}{4x^2-1} + 5.$

14. $\frac{x-5}{x+3} + \frac{x-8}{x-3} = \frac{80}{x^2-9} + \frac{1}{2}.$

15. $\frac{2x+1}{7-x} + \frac{4x+1}{7+x} = \frac{45}{49-x^2} + 1.$

Solution by Resolving into Factors.

297. A quadratic which has been reduced to its simplest form, and *has all its terms written on one side,* may often have that side resolved *by inspection* into factors, and the roots found by putting each factor equal to zero.

1. Solve $x^2 + 7x - 60 = 0$.

Since $x^2 + 7x - 60 = (x + 12)(x - 5)$, (§ 130)
the equation $x^2 + 7x - 60 = 0$
may be written $(x + 12)(x - 5) = 0$.

If *either* of the factors $x + 12$ or $x - 5$ is 0, the *product of the two factors* is 0, and the equation is satisfied.

Hence, $x + 12 = 0$, or $x - 5 = 0$.
$\therefore x = -12$, or $x = 5$.

2. Solve $x^3 - x^2 - 6x = 0$.

The equation $x^3 - x^2 - 6x = 0$
may be written $x(x^2 - x - 6) = 0$,
or $x(x - 3)(x + 2) = 0$, (§ 130)
and is satisfied if $x = 0, 3$, or -2.

Hence, the equation has *three* roots, 0, 3, -2.

3. Solve $x^3 - 2x^2 - 11x + 12 = 0$.

By the Factor Theorem (§ 135), we find that 1 put in place of x satisfies the equation, and is therefore a root of the equation.

Divide by $x - 1$, and resolve the quotient into its factors.

We have $(x - 1)(x - 4)(x + 3) = 0$.

Hence, the roots of the equation are 1, 4, -3.

4. Solve $x^2 + 3x - 10 = 0$.

If we add $\frac{9}{4}$, the square of half the coefficient of x, to the first two terms, we have a perfect trinomial square. Add and subtract $\frac{9}{4}$,

$$x^2+3x+\frac{9}{4}-\frac{9}{4}-10=0,$$

that is, $$\left(x+\frac{3}{2}\right)^2-\frac{49}{4}=0.$$

We now have the difference of two squares, and the factors are

$$\left(x+\frac{3}{2}+\frac{7}{2}\right)\left(x+\frac{3}{2}-\frac{7}{2}\right) \qquad (\S\ 126)$$

that is, $$(x+5)(x-2)=0.$$
$$\therefore x=-5, \text{ or } 2.$$

5. Solve $3x^2-2x-2=0$.

Divide by 3, $$x^2-\frac{2}{3}x-\frac{2}{3}=0.$$

Add and subtract the square of half the coefficient of x,

$$x^2-\frac{2}{3}x+\left(\frac{1}{3}\right)^2-\frac{1}{9}-\frac{2}{3}=0,$$

that is, $$\left(x-\frac{1}{3}\right)^2-\frac{7}{9}=0.$$

The square root of $\frac{7}{9}=\sqrt{\frac{1}{9}\times 7}=\frac{1}{3}\sqrt{7}$.

Hence, $$\left(x-\frac{1}{3}+\frac{1}{3}\sqrt{7}\right)\left(x-\frac{1}{3}-\frac{1}{3}\sqrt{7}\right)=0.$$

Therefore, $$x=\frac{1}{3}-\frac{1}{3}\sqrt{7}, \text{ or } \frac{1}{3}+\frac{1}{3}\sqrt{7}.$$

6. Solve $x^2-x+1=0$.

$$x^2-x+\frac{1}{4}-\frac{1}{4}+1=0.$$

$$\left(x-\frac{1}{2}\right)^2+\frac{3}{4}=0.$$

In order to make this the difference of two squares, write it

$$\left(x-\frac{1}{2}\right)^2-\left(-\frac{3}{4}\right).$$

The square root of $-\frac{3}{4}=\sqrt{\frac{1}{4}\times(-3)}=\frac{1}{2}\sqrt{-3}$.

Hence, $$\left(x-\frac{1}{2}+\frac{1}{2}\sqrt{-3}\right)\left(x-\frac{1}{2}-\frac{1}{2}\sqrt{-3}\right)=0.$$

Therefore, $$x=\frac{1}{2}-\frac{1}{2}\sqrt{-3}, \quad \text{or } x=\frac{1}{2}+\frac{1}{2}\sqrt{-3}.$$

Exercise 112.

Resolve into factors, and find the values of x:

1. $x^2 - 5x + 4 = 0$.   5. $x^3 + x^2 - 6x = 0$.

2. $6x^2 - 5x - 6 = 0$.   6. $x^3 - 8 = 0$.

3. $2x^2 - x - 3 = 0$.   7. $x^3 + 8 = 0$.

4. $10x^2 + x - 3 = 0$.   8. $x^4 - 16 = 0$.

9. $(x - 1)(x - 3)(x^2 + 5x + 6) = 0$.

10. $(2x - 1)(x - 2)(3x^2 - 5x - 2) = 0$.

11. $(x^2 + x - 2)(2x^2 + 3x - 5) = 0$.

12. $x^3 + x^2 - 4(x + 1) = 0$.

13. $3x^3 + 2x^2 - (3x + 2) = 0$.

14. $x^3 - 27 - 13(x - 3) = 0$.

15. $x^3 + 8 + 3(x^2 - 4) = 0$.   17. $2x^3 - 2x^2 - (x^2 - 1) = 0$.

16. $x(x^2 - 1) - 6(x - 1) = 0$.   18. $x^3 - 3x - 2 = 0$.

19. $2x^3 + 2x^2 + (x^2 - 5x - 6) = 0$.

298. Any **quadratic trinomial** of the form $ax^2 + bx + c$ can be resolved into two factors by writing it as *the difference of two squares.* Thus,

$$\begin{aligned} 3x^2 + 7x - 6 &= 3\left(x^2 + \tfrac{7}{3}x - 2\right) \\ &= 3\left[\left(x + \tfrac{7}{6}\right)^2 - \tfrac{121}{36}\right] \\ &= 3\left(x + \tfrac{7}{6} + \tfrac{11}{6}\right)\left(x + \tfrac{7}{6} - \tfrac{11}{6}\right) \\ &= 3(x + 3)\left(x - \tfrac{2}{3}\right) \\ &= (x + 3)(3x - 2). \end{aligned}$$

EXERCISE 113.

Resolve into factors:

1. $8x^2 - 26x + 21.$
2. $6x^2 - x - 7.$
3. $2x^2 + 7x + 6.$
4. $5x^2 + 26x + 24.$
5. $5x^2 - 15x + 11.$
6. $3x^2 + 5x - 2.$
7. $3x^2 - x - 1.$
8. $x^2 - 3x - 5.$

Literal Equations.

1. Solve the equation $adx - acx^2 = bcx - bd.$

Transpose bcx and change the signs,

$$acx^2 + bcx - adx = bd.$$

Express the left member in *two terms*,

$$acx^2 + (bc - ad)x = bd.$$

Multiply by 4 times the coefficient of x^2,

$$4a^2c^2x^2 + 4ac(bc - ad)x = 4abcd.$$

Complete the square,

$$4a^2c^2x^2 + (\quad) + (bc - ad)^2 = b^2c^2 + 2abcd + a^2d^2.$$

Extract the root, $\quad 2acx + (bc - ad) = \pm(bc + ad).$

Reduce, $\quad 2acx = -(bc - ad) \pm (bc + ad)$

$$= 2ad, \text{ or } -2bc.$$

$$\therefore x = \frac{d}{c}, \text{ or } -\frac{b}{a}.$$

2. Solve the equation $px^2 - px + qx^2 + qx = \dfrac{pq}{p+q}.$

$$(p+q)x^2 - (p-q)x = \frac{pq}{p+q}.$$

$$4(p+q)^2x^2 - 4(p^2 - q^2)x = 4pq.$$

$$4(p+q)^2x^2 - (\quad) + (p-q)^2 = p^2 + 2pq + q^2.$$

$$2(p+q)x - (p-q) = \pm(p+q).$$

$$2(p+q)x = (p-q) \pm (p+q).$$

$$\therefore x = \frac{p}{p+q}, \text{ or } -\frac{q}{p+q}.$$

NOTE. The left-hand member of the equation when simplified must be expressed in *two terms, simple or compound*, one term containing x^2 and the other term containing x.

EXERCISE 114.

Solve:

1. $x^2 + 2ax = 3a^2.$
2. $x^2 - 4ax = 12a^2.$
3. $x^2 + 8bx = 9b^2.$
4. $x^2 + 3bx = 10b^2.$
5. $x^2 + 5ax = 14a^2.$
6. $3x^2 + 4cx = 4c^2.$
7. $5ax - 2x^2 = 2a^2.$
8. $6x^2 - ax - a^2 = 0.$
9. $2a^2x^2 + ax - 1 = 0.$
10. $12b^2x^2 - 5bx = 3.$
11. $\dfrac{2x^2}{3} + \dfrac{ax}{4} = 11a(x - 3a).$
12. $\dfrac{3x^2}{4} + \dfrac{x}{2a} = \dfrac{3}{2a^2}.$
13. $x^2 - \dfrac{x}{a} = \dfrac{3}{4a^2}.$
14. $\dfrac{3ax^2}{4} + \dfrac{2x}{3} = \dfrac{13}{3a}.$
15. $\dfrac{x^2 + ax + a^2}{3} + \dfrac{a^2 - x^2}{4} = ax.$
16. $2x^2 - x + a = 2a^2.$
17. $\dfrac{x(a - x)}{a + x} + \dfrac{x}{3} = 5a.$
18. $\dfrac{x(3x - a)}{4x + a} = \dfrac{a}{12}.$
19. $x^2 + \dfrac{m^2 - n^2}{mn}x = 1.$
20. $\dfrac{x^2}{6b^2} - \dfrac{5x}{6ab} + \dfrac{1}{a^2} = 0.$
21. $\dfrac{x + a}{b - a} + \dfrac{b - a}{x + a} = 2.$
22. $x^2 + (a - b)x = ab.$
23. $\dfrac{2a + x}{2a - x} + \dfrac{a - 2x}{a + 2x} = \dfrac{8}{3}.$
24. $\dfrac{2x - 3a}{x + 4a} + \dfrac{3x + 2a}{4x - a} = \dfrac{10}{7}.$
25. $\dfrac{a + 2b}{3b - x} + \dfrac{a + b}{x + 2a} = 2.$
26. $\dfrac{a - b + x}{a + b + x} + \dfrac{a + b}{x + b} = 2.$
27. $\dfrac{1}{a + b + x} = \dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{x}.$
28. $9x^2 - 3(a + 2b)x + 2ab = 0.$
29. $(2a + 1)x^2 + 3a^2x + a^3 - a^2 = 0.$
30. $(1 - a^2)x^2 - 2(1 + a^2)x + 1 - a^2 = 0.$
31. $(a + b)^2x^2 - (a^2 - b^2)x = ab.$
32. $(a + b)x^2 - (2a + b)x + a = 0.$

33. $\frac{1}{x}+\frac{1}{x+b}=\frac{1}{a}+\frac{1}{a+b}.$

34. $\frac{1}{a-x}-\frac{1}{a+x}=\frac{3+x^2}{a^2-x^2}.$

35. $\frac{x^2+2ab(a^2+b^2)}{a^2+b^2}=2x.$

36. $\frac{(2x-a)^2}{2x-a+2b}=b.$

37. $\frac{(a-1)^2x^2+2(3a-1)x}{4a-1}=1.$

38. $\frac{a}{x-a}+\frac{b}{x-b}=\frac{2c}{x-c}.$

39. $cx=ax^2+bx^2-\frac{ac}{a+b}.$

40. $\frac{x+3}{x-3}=a+\frac{x-3}{x+3}.$

41. $mx^2-1=\frac{x(m^3-n^2)}{mn}.$

42. $(ax-b)(bx-a)=c^2.$

43. $\frac{ax+b}{bx+a}=\frac{mx+n}{nx+m}.$

44. $\frac{m}{m+x}+\frac{m}{m-x}=c.$

45. $\frac{(a^2-b^2)(x^2+1)}{a^2+b^2}=2x.$

46. $\frac{x^2-4mnx}{(m+n)^2}=(m-n)^2.$

47. $\frac{8}{9}(x^2+a^2+ab)=\frac{1}{9}x(20a+4b).$

48. $x^2-2mx=(n-p+m)(n-p-m).$

49. $x^2-(m+n)x=\frac{1}{4}(p+q+m+n)(p+q-m-n).$

50. $mnx^2-(m+n)(mn+1)x+(m+n)^2=0.$

51. $\frac{x^2}{3m-2a}-\frac{m^2-4a^2}{4a-6m}=\frac{x}{2}.$

52. $6x+\frac{(a+b)^2}{x}=5(a-b)+\frac{25ab}{6x}.$

53. $\frac{2b-x-2a}{bx}+\frac{4b-7a}{ax-bx}=\frac{x-4a}{ab-b^2}.$

54. $\frac{a-2b-x}{a^2-4b^2}-\frac{5b-x}{ax+2bx}+\frac{2a-x-19b}{2bx-ax}=0.$

55. $\frac{x+13a+3b}{5a-3b-x}-1=\frac{a-2b}{x+2b}.$

56. $\frac{x+3b}{8a^2-12ab}-\frac{3b}{9b^2-4a^2}-\frac{a+3b}{(2a+3b)(x-3b)}=0.$

Solution by a Formula.

299. Every affected quadratic can be reduced to the form

$$ax^2 + bx + c = 0.$$

Solve $ax^2 + bx + c = 0$.

Transpose c, $ax^2 + bx = -c$.

Multiply the equation by $4a$ and add the square of b,

$$4a^2x^2 + (\quad) + b^2 = b^2 - 4ac.$$

Extract the root, $2ax + b = \pm\sqrt{b^2 - 4ac}$.

$$\therefore x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

By this formula, the values of x in an equation of the form $ax^2 + bx + c = 0$ may be written at once.

Solve $3x^2 - 5x + 2 = 0$.

Here $a = 3$, $b = -5$, $c = 2$.

Putting these values for the letters in the above formula, we have

$$x = \frac{5 + \sqrt{25 - 24}}{6}, \text{ or } \frac{5 - \sqrt{25 - 24}}{6}$$
$$= \tfrac{6}{6}, \text{ or } \tfrac{4}{6}$$
$$= 1, \text{ or } \tfrac{2}{3}.$$

EXERCISE 115.

Solve by the above formula:

1. $2x^2 + 3x = 14$.
2. $3x^2 - 5x = 12$.
3. $x^2 - 7x = 18$.
4. $5x^2 - x = 42$.
5. $6x^2 - 7x = 10$.
6. $3x^2 - 11x = -6$.
7. $5x^2 - 7x = -2$.
8. $4x^2 - 9x = 28$.
9. $5x^2 + 7x = 12$.
10. $11x^2 - 9x = -\frac{10}{9}$.
11. $7x^2 + 5x = 38$.
12. $5x^2 - 7x = 6$.

Equations Involving Two or More Radicals.

300. Solve $\sqrt{x+4}+\sqrt{2x+6}=\sqrt{7x+14}$.

Square, $x+4+2\sqrt{(x+4)(2x+6)}+2x+6=7x+14$.

Simplify, $\sqrt{(x+4)(2x+6)}=2x+2$.

Square, $(x+4)(2x+6)=(2x+2)^2$.

Solve, $x=5$, or -2.

Of these two values, only 5 will satisfy the original equation.

Squaring both members of the original equation is equivalent to transposing $\sqrt{7x+14}$ to the left member, and then multiplying by the rationalizing factor $\sqrt{x+4}+\sqrt{2x+6}+\sqrt{7x+14}$, so that

$$(\sqrt{x+4}+\sqrt{2x+6}-\sqrt{7x+14})(\sqrt{x+4}+\sqrt{2x+6}+\sqrt{7x+14})=0,$$

and this reduces to $\sqrt{(x+4)(2x+6)}-(2x+2)=0$.

Transposing and squaring again is equivalent to multiplying by

$$(\sqrt{x+4}-\sqrt{2x+6}+\sqrt{7x+14})(\sqrt{x+4}-\sqrt{2x+6}-\sqrt{7x+14}).$$

Reducing, $x^2-3x-10=0$.

Therefore, the equation $x^2-3x-10=0$ is really obtained from

$$\begin{aligned}&(\sqrt{x+4}+\sqrt{2x+6}-\sqrt{7x+14})\\ \times&(\sqrt{x+4}+\sqrt{2x+6}+\sqrt{7x+14})\\ \times&(\sqrt{x+4}-\sqrt{2x+6}-\sqrt{7x+14})\\ \times&(\sqrt{x+4}-\sqrt{2x+6}+\sqrt{7x+14})=0.\end{aligned}$$

This last equation is satisfied by any value that will satisfy any one of the *four* factors of its left member. The first factor is satisfied by 5, and the last factor by -2, while no values can be found to satisfy the second or third factor.

As 5 is the only value of x that will satisfy the *original equation*, all other values must be rejected.

EXERCISE 116.

Solve:

1. $\sqrt{9x+40} - 2\sqrt{x+7} = \sqrt{x}$.

2. $\sqrt{a+x} + \sqrt{a-x} = \sqrt{b}$.

3. $\dfrac{3x+\sqrt{4x-x^2}}{3x-\sqrt{4x-x^2}} = 2$.

4. $\sqrt{x-3} - \sqrt{x-14} = \sqrt{4x-155}$.

5. $\sqrt{x+4} - \sqrt{x} = \sqrt{x+\frac{3}{2}}$.

6. $\dfrac{3\sqrt{x}-4}{2+\sqrt{x}} - \dfrac{15+3\sqrt{x}}{40+\sqrt{x}} = 0$.

7. $\sqrt{14x+9} + 2\sqrt{x+1} + \sqrt{3x+1} = 0$.

8. $\sqrt{5x+1} - 2 - \sqrt{x+1} = 0$.

9. $\sqrt{x-2} + \sqrt{x+3} - \sqrt{4x+1} = 0$.

10. $\sqrt{7-x} + \sqrt{3x+10} + \sqrt{x+3} = 0$.

11. $3\sqrt{x^3+17} + \sqrt{x^3+1} + 2\sqrt{5x^3+41} = 0$.

12. $2x - \sqrt{2x-1} = x+2$.

13. $\sqrt{x+2} - \sqrt{x-2} - \sqrt{2x} = 0$.

14. $\dfrac{1}{x+\sqrt{x^2-1}} + \dfrac{1}{x-\sqrt{x^2-1}} = 12$.

15. $\sqrt{3x} + \sqrt{3x+13} = \dfrac{91}{\sqrt{3x+13}}$.

16. $x - \sqrt[3]{x^3-2x^2} - 2 = 0$.

17. $\dfrac{3x-\sqrt{x^2-8}}{x-\sqrt{x^2-8}} = x + \sqrt{x^2-8}$.

Equations in the Quadratic Form.

301. An equation is in the *quadratic form* if it contains but two powers of the unknown number, and the exponent of one power is exactly twice that of the other power.

302. Equations in the quadratic form may be solved by the methods for solving quadratics.

1. Solve $8x^6 + 63x^3 = 8$.

Multiply by 32 and complete the square,

$$256x^6 + (\quad) + 63^2 = 4225.$$

Extract the square root, $16x^3 + 63 = \pm 65.$

Hence, $x^3 = \frac{1}{8}, \text{ or } -8.$

Extracting the cube root, two values of x are $\frac{1}{2}$ and -2. There are four other values of x which may be found by § 297.

2. Solve $\sqrt{x^3} - 3\sqrt[4]{x^3} = 40$.

Using fractional exponents we have,

$$x^{\frac{3}{2}} - 3x^{\frac{3}{4}} = 40.$$

Complete the square, $4x^{\frac{3}{2}} - 12x^{\frac{3}{4}} + 9 = 169.$

Extract the root, $2x^{\frac{3}{4}} - 3 = \pm 13.$

Transpose -3, $2x^{\frac{3}{4}} = 16, \text{ or } -10.$

Divide by 2, $x^{\frac{3}{4}} = 8, \text{ or } -5.$

Extract the cube root, $x^{\frac{1}{4}} = 2, \text{ or } -5^{\frac{1}{3}}.$

Raise to the fourth power, $x = 16, \text{ or } 5\sqrt[3]{5}.$

3. Solve $(2x - 3)^2 - (2x - 3) = 6$.

Put y for $2x - 3$, and therefore y^2 for $(2x - 3)^2$.

We have $y^2 - y = 6.$

Solving, $y = 3, \text{ or } -2.$

Put $2x - 3$ for y,

$$2x - 3 = 3, \qquad\qquad 2x - 3 = -2.$$

$$x = 3. \qquad\qquad x = \tfrac{1}{2}.$$

4. Solve $7x^2 - 5x + 8\sqrt{7x^2 - 5x + 1} = -8$.

By adding 1 to both sides, we have

$$7x^2 - 5x + 1 + 8\sqrt{7x^2 - 5x + 1} = -7.$$

Put y for $\sqrt{7x^2 - 5x + 1}$, and hence y^2 for $7x^2 - 5x + 1$.

Then, $y^2 + 8y = -7$.

Solving, $y = -1$, or -7.

Therefore, $y^2 = 1$, or 49.

We now have $7x^2 - 5x + 1 = 1$, or $7x^2 - 5x + 1 = 49$.

Solving these equations, we find for the values of x,

$$0, \frac{5}{7}; \text{ or } 3, -\frac{16}{7}.$$

These values all satisfy the given equation when we take the *negative* value of the square root of the expression $7x^2 - 5x + 1$; they are in fact the four roots of the biquadratic obtained by clearing the given equation of radicals.

5. Solve $x^4 - 10x^3 + 35x^2 - 50x + 24 = 0$.

Take the square root of the left side.

$$
\begin{array}{rl}
 & x^4 - 10x^3 + 35x^2 - 50x + 24 \,\underline{|\, x^2 - 5x + 5} \\
 & x^4 \\
2x^2 - 5x & \overline{\left| -10x^3 + 35x^2 \right.} \\
 & \underline{\left| -10x^3 + 25x^2 \right.} \\
2x^2 - 10x + 5 & \left| 10x^2 - 50x + 24 \right. \\
 & \underline{\left| 10x^2 - 50x + 25 \right.} \\
 & \hphantom{10x^2 - 50x +} -1
\end{array}
$$

It is now seen that if 1 is added, the square will be complete and the equation will be

$$x^4 - 10x^3 + 35x^2 - 50x + 25 = 1.$$

Extract the square root, and the result is

$$x^2 - 5x + 5 = \pm 1.$$

Solving, $x = 4, 1, 3$, or 2.

EXERCISE 117.

Solve:

1. $x^4 - 5x^2 + 4 = 0.$
2. $x^4 - 13x^2 + 36 = 0.$
3. $x^4 - 21x^2 = 100.$
4. $4x^6 - 3x^3 = 27.$
5. $2x^4 + 5x^2 = 21\frac{3}{8}.$
6. $10x^4 - 21 = x^2.$
7. $\sqrt[3]{x^2} + 3\sqrt[3]{x} = 1\frac{3}{4}.$
8. $3\sqrt[4]{x} - 2\sqrt{x} = -20.$
9. $5x^{2n} + 3x^n = 6\frac{3}{4}.$
10. $(8x+3)^2 + (8x+3) = 30.$
11. $2(x^2 - x + 1) - \sqrt{x^2 - x + 1} = 1.$
12. $x^6 - 9x^3 + 8 = 0.$
13. $x^{\frac{1}{2}} + x^{-\frac{1}{2}} = \dfrac{13}{6}.$
14. $(x+1) + \sqrt{x+1} = 6.$
15. $x^4 - 13x^2 = -36.$
16. $2x^2 + 4x + 9 + 3\sqrt{2x^2 + 4x + 9} = 40.$
17. $2x^2 + 3x - 5\sqrt{2x^2 + 3x + 9} = -3.$
18. $3x^2 + 15x - 2\sqrt{x^2 + 5x + 1} = 2.$
19. $x^2 - \frac{3}{2}x + 3\sqrt{2x^2 - 3x + 2} = 7.$
20. $2x^2 - \sqrt{x^2 - 2x - 3} = 4x + 9.$
21. $3x^2 - 4x + \sqrt{3x^2 - 4x - 6} = 18.$
22. $3x^2 - 7 + 3\sqrt{3x^2 - 16x + 21} = 16x.$
23. $x^4 - 2x^3 - 13x^2 + 14x + 24 = 0.$
24. $x^4 - 4x^3 - 10x^2 + 28x - 15 = 0.$
25. $4x^4 - 20x^3 + 23x^2 + 5x - 6 = 0.$
26. $4x^4 - 12x^3 + 5x^2 + 6x - 15 = 0.$
27. $4x^4 - 12x^3 + 17x^2 - 12x - 12 = 0.$
28. $6x^2 + 6x + \sqrt{x(x+1)} = 7.$

Character of the Roots.

The two roots of $ax^2 + bx + c = 0$ are

$$-\frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} \text{ and } -\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}. \qquad (\S\ 299)$$

303. The *character* of the two roots depends wholly upon $b^2 - 4ac$, the *expression under the radical sign.*

1. **If $b^2 - 4ac$ is positive and not zero,** the roots are *real* and *unequal.*

The roots are real, since the *square root* of the *positive number,* $b^2 - 4ac$, can be found exactly or approximately.

The roots are unequal, since $\sqrt{b^2 - 4ac}$ is not zero.

If $b^2 - 4ac$ is a perfect square, the roots are *rational;* if $b^2 - 4ac$ is not a perfect square, the roots are *surds.*

2. **If $b^2 - 4ac$ is zero,** the two roots are *real* and *equal,* since they both become $-\frac{b}{2a}$. Hence,

The roots of $ax^2 + bx + c = 0$ are *real,* if $b^2 =$ or $> 4ac$.

3. **If $b^2 - 4ac$ is negative,** the roots are *imaginary,* since they involve the *square root* of a *negative number.* Hence,

The roots of $ax^2 + bx + c = 0$ are *imaginary,* if $b^2 < 4ac$.

The two imaginary roots of a quadratic cannot be equal, since $b^2 - 4ac$ is not zero. They have, however, the same real parts, and the same imaginary parts with opposite signs, and are, therefore, conjugate imaginaries, § 284.

The expression $b^2 - 4ac$ is called the **discriminant** of the expression $ax^2 + bx + c$.

304. The above cases may be summarized as follows:

CASE 1. If $b^2 - 4ac > 0$, the roots are real and unequal.
CASE 2. If $b^2 - 4ac = 0$, the roots are real and equal.
CASE 3. If $b^2 - 4ac < 0$, the roots are imaginary.

305. By finding the value of $b^2 - 4ac$ we can determine at once the character of the roots of a given equation.

1. $x^2 - 5x + 6 = 0$.

Here $$a = 1,\ b = -5,\ c = 6.$$
$$b^2 - 4ac = 25 - 24 = 1.$$

The roots are real and unequal, and rational.

2. $3x^2 + 7x - 1 = 0$.

Here $$a = 3,\ b = 7,\ c = -1.$$
$$b^2 - 4ac = 49 + 12 = 61.$$

The roots are real and unequal, and are both surds.

3. $4x^2 - 12x + 9 = 0$.

Here $$a = 4,\ b = -12,\ c = 9.$$
$$b^2 - 4ac = 144 - 144 = 0.$$

The roots are real and equal.

4. $2x^2 - 3x + 4 = 0$.

Here $$a = 2,\ b = -3,\ c = 4.$$
$$b^2 - 4ac = 9 - 32 = -23.$$

The roots are both imaginary.

5. Find the values of m for which

$$2mx^2 + (5m + 2)x + (4m + 1) = 0$$

has its two roots equal.

Here $$a = 2m,\ b = 5m + 2,\ c = 4m + 1.$$

If the roots are to be equal, we must have

$$b^2 - 4ac = 0, \text{ or } (5m + 2)^2 - 8m(4m + 1) = 0.$$

Solving, $$m = 2, \text{ or } -\frac{2}{7}.$$

For these values of m the equation becomes

$$4x^2 + 12x + 9 = 0, \text{ and } 4x^2 - 4x + 1 = 0,$$

each of which has its roots equal.

EXERCISE 118.

Determine without solving the character of the roots of each of the following equations:

1. $x^2 + 5x + 6 = 0.$
2. $x^2 + 2x - 15 = 0.$
3. $x^2 + 2x + 3 = 0.$
4. $3x^2 + 7x + 2 = 0.$
5. $9x^2 + 6x + 1 = 0.$
6. $6x^2 - 7x - 3 = 0.$
7. $5x^2 - 5x - 3 = 0.$
8. $2x^2 - x + 5 = 0.$
9. $6x^2 + x - 77 = 0.$
10. $5x^2 + 8x + \frac{16}{5} = 0.$

Determine the values of m for which the two roots of each of the following equations are equal:

11. $(m + 1)x^2 + (m - 1)x + m + 1 = 0.$
12. $(2m - 3)x^2 + mx + m - 1 = 0.$
13. $2mx^2 + x^2 + 4x + 2mx + 2m - 4 = 0.$
14. $2mx^2 + 3mx - 6 = 3x - 2m - x^2.$
15. $mx^2 + 9x - 10 = 3mx - 2x^2 + 2m.$

Relations of Roots and Coefficients.

306. If we divide the general equation $ax^2 + bx + c = 0$ by a, we have the equation $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$; this may be written $x^2 + px + q = 0$, where $p = \frac{b}{a}$, $q = \frac{c}{a}\cdot$

307. By solving $x^2 + px + q = 0$, and denoting the first value of x by r_1, and the second value by r_2, we have

$$r_1 = -\frac{p}{2} + \frac{\sqrt{p^2 - 4q}}{2},$$

$$r_2 = -\frac{p}{2} - \frac{\sqrt{p^2 - 4q}}{2}.$$

Add, $\quad r_1 + r_2 = -p.$

Multiply, $\quad r_1 r_2 = q.$

It appears, then, that if any quadratic equation is made to assume the form $x^2 + px + q = 0$, the following relations hold between the coefficients and roots of the equation:

1. The **sum** of the two roots is equal to the coefficient of x with its sign changed.

2. The **product** of the two roots is equal to the constant term.

308. If r_1 and r_2 are the roots of the equation $x^2 + px + q = 0$, the equation may be written

$$(x - r_1)(x - r_2) = 0.$$

309. Form the equation of which the roots are 3 and -2.

The equation is $\quad (x - 3)(x + 2) = 0,$

or $\quad x^2 - x - 6 = 0.$

EXERCISE 119.

Form the equation of which the roots are:

1. $7, 6.$
2. $5, -3.$
3. $1\frac{1}{2}, -2.$
4. $4, 2\frac{1}{3}.$
5. $1\frac{1}{2}, -1\frac{1}{3}.$
6. $-1\frac{1}{6}, -1\frac{2}{3}.$
7. $13, -4\frac{1}{3}.$
8. $\frac{3}{11}, \frac{2}{11}.$
9. $3 + \sqrt{2}, 3 - \sqrt{2}.$
10. $1 + \sqrt{-1}, 1 - \sqrt{-1}$
11. $a, a - b.$
12. $a + b, a - b.$

Problems Involving Quadratics.

310. Problems that involve quadratic equations apparently have two solutions, since a quadratic equation has two roots. If both roots of the quadratic equation are positive integers, they will, generally, both be admissible solutions.

Fractional and negative roots will in some problems give admissible solutions; in other problems they will not.

No difficulty will be found in selecting the result which belongs to the particular problem we are solving. Sometimes, by a change in the statement of the problem, we may form a new problem which corresponds to the result that was inapplicable to the original problem.

Imaginary roots indicate that the problem is impossible.

Here, as in simple equations, x stands for an unknown *number*.

1. The sum of the squares of two consecutive numbers is 481. Find the numbers.

Let $x =$ one number,
and $x + 1 =$ the other.
Then, $x^2 + (x + 1)^2 = 481,$
or $2x^2 + 2x + 1 = 481.$

The solution of which gives $x = 15$, or -16.

The positive root 15 gives for the numbers, 15 and 16.

The negative root -16 is inapplicable to the problem, as *consecutive numbers* are understood to be integers which follow one another in the common scale, 1, 2, 3, 4 ·····

2. A pedler bought a number of knives for $2.40. Had he bought 4 more for the same money, he would have paid 3 cents less for each. How many knives did he buy, and what did he pay for each?

Let $x =$ number of knives he bought.
Then, $\frac{240}{x} =$ number of cents he paid for each.

But if $x + 4 =$ number of knives he bought,

$\frac{240}{x+4} =$ number of cents he paid for each,

$\frac{240}{x} - \frac{240}{x+4} =$ the difference in price.

But $3 =$ the difference in price.

$$\therefore \frac{240}{x} - \frac{240}{x+4} = 3.$$

Solving, $x = 16$, or -20.

He bought 16 knives, therefore, and paid $\frac{240}{16}$, or 15 cents for each.

If the problem is changed so as to read: A pedler bought a number of knives for $2.40; if he had bought 4 *less* for the same money, he would have paid 3 cents *more* for each, the equation will be

$$\frac{240}{x-4} - \frac{240}{x} = 3.$$

Solving, $x = 20$, or -16.

This second problem is therefore the one which the negative answer of the first problem suggests.

3. What is the price of eggs per dozen when 2 more in a shilling's worth lowers the price 1 penny per dozen?

Let $x =$ number of eggs for a shilling.

Then, $\frac{1}{x} =$ cost of 1 egg in shillings,

and $\frac{12}{x} =$ cost of 1 dozen in shillings.

But if $x + 2 =$ number of eggs for a shilling,

$\frac{12}{x+2} =$ cost of 1 dozen in shillings.

$$\therefore \frac{12}{x} - \frac{12}{x+2} = \frac{1}{12} \text{ (1 penny being } \tfrac{1}{12} \text{ of a shilling).}$$

The solution of which gives $x = 16$, or -18.

And, if 16 eggs cost a shilling, 1 dozen will cost 9 pence.

Therefore, the price of the eggs is 9 pence per dozen

If the problem is changed so as to read: What is the price of eggs per dozen when two *less* in a shilling's worth *raises* the price 1 penny per dozen? the equation will be

$$\frac{12}{x-2}-\frac{12}{x}=\frac{1}{12}.$$

The solution of which gives $x = 18$, or -16.

Hence, the number 18, which had a negative sign and was inapplicable in the original problem, is here the true result.

EXERCISE 120.

1. The sum of two squares of two consecutive integers is 761. Find the numbers.

2. The sum of the squares of two consecutive numbers exceeds the product of the numbers by 13. Find the numbers.

3. The square of the sum of two consecutive even numbers exceeds the sum of their squares by 336. Find the numbers.

4. Twice the product of two consecutive numbers exceeds the sum of the numbers by 49. Find the numbers.

5. The sum of the squares of three consecutive numbers is 110. Find the numbers.

6. The difference of the cubes of two consecutive odd numbers is 602. Find the numbers.

7. The length of a rectangular field exceeds its breadth by 2 rods. If the length and breadth of the field were each increased by 4 rods, the area would be 80 square rods. Find the dimensions of the field.

8. The area of a square may be doubled by increasing its length by 10 feet and its breadth by 3 feet. Find the length of its side.

9. A rectangular grass plot 12 yards long and 9 yards wide has a path around it. The area of the path is $\frac{2}{3}$ of the area of the plot. Find the width of the path.

10. The perimeter of a rectangular field is 60 rods. Its area is 200 square rods. Find its dimensions.

11. The length of a rectangular plot is 10 rods more than twice its width, and the length of a diagonal of the plot is 25 rods. What are the dimensions of the plot?

12. The denominator of a certain fraction exceeds the numerator by 3. If both numerator and denominator are increased by 4, the fraction will be increased by $\frac{1}{8}$. Find the fraction.

13. The numerator of a fraction exceeds twice the denominator by 1. If the numerator is decreased by 3, and the denominator increased by 3, the resulting fraction will be the reciprocal of the given fraction. Find the fraction.

14. A farmer sold a number of sheep for $120. If he had sold 5 less for the same money, he would have received $2 more a sheep. How much did he receive a sheep?

State the problem to which the negative solution applies.

15. A merchant sold a certain number of yards of silk for $40.50. If he had sold 9 yards more for the same money, he would have received 75 cents less per yard. How many yards did he sell?

16. A man bought a number of geese for $27. He sold all but two for $25, thus gaining 25 cents on each goose sold. How many geese did he buy?

17. A man agrees to do a piece of work for $48. It takes him 4 days longer than he expected, and he finds that he has earned $1 less per day than he expected. In how many days did he expect to do the work?

18. Find the price of eggs per dozen when 10 more in one dollar's worth lowers the price 4 cents a dozen.

19. A man sold a horse for $171, and gained as many per cent on the sale as the horse cost dollars. How much did the horse cost?

20. A drover bought a certain number of sheep for $160. He kept four, and sold the remainder for $10.60 per head, and made on his investment $\frac{3}{4}$ as many per cent as he paid dollars for each sheep bought. How many sheep did he buy?

21. Two pipes running together can fill a cistern in $5\frac{5}{6}$ hours. The larger pipe will fill the cistern in 4 hours less time than the smaller. How long will it take each pipe running alone to fill the cistern?

22. A and B can do a piece of work together in 18 days, and it takes B 15 days longer to do it alone than it does A. In how many days can each do it alone?

23. A boat's crew row 4 miles down a river and back again in 1 hour and 30 minutes. Their rate in still water is 2 miles an hour faster than twice the rate of the current. Find the rate of the crew and the rate of the current.

24. A number is formed by two digits. The units' digit is 2 more than the square of half the tens' digit, and if 18 is added to the number, the order of the digits will be reversed. Find the number.

25. A circular grass plot is surrounded by a path of a uniform width of 3 feet. The area of the path is $\frac{7}{9}$ the area of the plot. Find the radius of the plot.

26. If a carriage wheel 11 feet round took $\frac{1}{4}$ of a second less to revolve, the rate of the carriage would be five miles more per hour. At what rate is the carriage traveling?

CHAPTER XX.

SIMULTANEOUS QUADRATICS.

311. Quadratic equations involving two unknown numbers require different methods for their solution, according to the form of the equations.

CASE 1.

312. When one of the equations is a simple equation.

Solve
$$\left.\begin{aligned} 3x^2 - 2xy &= 5 \\ x - y &= 2 \end{aligned}\right\} \qquad \begin{aligned} &(1) \\ &(2) \end{aligned}$$

Transpose x in (2), $y = x - 2$.

In (1) put $x - 2$ for y,

$$3x^2 - 2x(x - 2) = 5.$$

The solution of which gives $x = 1$, or $x = -5$.

If $x = 1$,

$y = 1 - 2 = -1$;

and if $x = -5$,

$y = -5 - 2 = -7$.

We have, therefore, the *pairs* of values,

$$\left.\begin{aligned} x &= 1 \\ y &= -1 \end{aligned}\right\}; \text{ or } \left.\begin{aligned} x &= -5 \\ y &= -7 \end{aligned}\right\}.$$

The original equations are both satisfied by either pair of values. But the values $x = 1$, $y = -7$, will not satisfy the equations; nor will the values $x = -5$, $y = -1$.

The student must be careful to join to each value of x *the corresponding value of* y.

CASE 2.

313. When the left side of each of the two equations is homogeneous and of the second degree.

Solve
$$\left.\begin{aligned} 2y^2 - 4xy + 3x^2 &= 17 \\ y^2 - x^2 &= 16 \end{aligned}\right\} \qquad \begin{aligned}(1)\\(2)\end{aligned}$$

Let $y = vx$ and substitute vx for y in both equations.

From (1), $$2v^2x^2 - 4vx^2 + 3x^2 = 17.$$

$$\therefore x^2 = \frac{17}{2v^2 - 4v + 3}.$$

From (2), $$v^2x^2 - x^2 = 16.$$

$$\therefore x^2 = \frac{16}{v^2 - 1}.$$

Equate the values of x^2, $\dfrac{17}{2v^2 - 4v + 3} = \dfrac{16}{v^2 - 1}$,

$$32v^2 - 64v + 48 = 17v^2 - 17,$$

$$15v^2 - 64v = -65,$$

$$225v^2 - 960v = -975,$$

$$225v^2 - (\ ) + 32^2 = 49,$$

$$15v - 32 = \pm 7.$$

$$\therefore v = \frac{5}{3}, \text{ or } \frac{13}{5}.$$

If $$v = \frac{5}{3},$$

$$y = vx = \frac{5x}{3}.$$

Substitute in (2),

$$\frac{25x^2}{9} - x^2 = 16,$$

$$x^2 = 9,$$

$$x = \pm 3,$$

$$y = \frac{5x}{3} = \pm 5.$$

If $$v = \frac{13}{5},$$

$$y = vx = \frac{13x}{5}.$$

Substitute in (2),

$$\frac{169x^2}{25} - x^2 = 16,$$

$$x^2 = \frac{25}{9},$$

$$x = \pm\frac{5}{3},$$

$$y = \frac{13x}{5} = \pm\frac{13}{3}.$$

CASE 3.

314. When the two equations are symmetrical with respect to x and y; that is, when x and y are similarly involved.

Thus, the expressions

$$2x^3+3x^2y^2+2y^3,\ 2xy-3x-3y+1,\ x^4-3x^2y-3xy^2+y^4$$

are symmetrical expressions. In this case the general rule is to combine the equations in such a manner as to remove the highest powers of x and y.

Solve
$$\left.\begin{aligned} x^4+y^4&=337 \\ x+y&=7 \end{aligned}\right\} \qquad \begin{aligned}(1)\\(2)\end{aligned}$$

To remove x^4 and y^4, raise (2) to the fourth power,

$$\begin{array}{lr} & x^4+4x^3y+6x^2y^2+4xy^3+y^4=2401 \\ \text{Add (1),} & x^4 \qquad\qquad\qquad\qquad\qquad +y^4=\ \ 337 \\ \hline & 2x^4+4x^3y+6x^2y^2+4xy^3+2y^4=2738 \end{array}$$

Divide by 2, $x^4+2x^3y+3x^2y^2+2xy^3+y^4=1369.$

Extract the square root, $x^2+xy+y^2=\pm 37.$ (3)

Subtract (3) from $(2)^2$, $xy=12$ or 86.

We now have to solve the two pairs of equations,

$$\left.\begin{aligned} x+y&=7 \\ xy&=12 \end{aligned}\right\};\quad \left.\begin{aligned} x+y&=7 \\ xy&=86 \end{aligned}\right\}.$$

From the first,
$$\left.\begin{aligned} x&=4 \\ y&=3 \end{aligned}\right\};\ \text{or}\ \left.\begin{aligned} x&=3 \\ y&=4 \end{aligned}\right\}.$$

From the second,
$$\left.\begin{aligned} x&=\frac{7\pm\sqrt{-295}}{2} \\ y&=\frac{7\mp\sqrt{-295}}{2} \end{aligned}\right\}.$$

315. The preceding cases are *general methods* for the solution of equations that belong to the kinds referred to; often, however, in the solution of these and other kinds of simultaneous equations involving quadratics, a little ingenuity will suggest some step by which the roots may be found more easily than by the general method.

1. Solve
$$\left.\begin{aligned} x+y&=40 \\ xy&=300 \end{aligned}\right\} \qquad \begin{aligned}&(1)\\&(2)\end{aligned}$$

Square (1), $$x^2+2xy+y^2=1600. \qquad (3)$$

Multiply (2) by 4, $$4xy=1200. \qquad (4)$$

Subtract (4) from (3),
$$x^2-2xy+y^2=400. \qquad (5)$$

Extract the square root, $$x-y=\pm 20. \qquad (6)$$

From (1) and (6), $$\left.\begin{aligned} x&=30 \\ y&=10 \end{aligned}\right\};\ \text{or}\ \left.\begin{aligned} x&=10 \\ y&=30 \end{aligned}\right\}.$$

2. Solve
$$\left.\begin{aligned} \frac{1}{x}+\frac{1}{y}&=\frac{9}{20} \\ \frac{1}{x^2}+\frac{1}{y^2}&=\frac{41}{400} \end{aligned}\right\} \qquad \begin{aligned}&(1)\\&(2)\end{aligned}$$

Square (1), $$\frac{1}{x^2}+\frac{2}{xy}+\frac{1}{y^2}=\frac{81}{400}. \qquad (3)$$

Subtract (2) from (3), $$\frac{2}{xy}=\frac{40}{400}. \qquad (4)$$

Subtract (4) from (2),
$$\frac{1}{x^2}-\frac{2}{xy}+\frac{1}{y^2}=\frac{1}{400}.$$

Extract the square root, $$\frac{1}{x}-\frac{1}{y}=\pm\frac{1}{20}. \qquad (5)$$

From (1) and (5), $$\left.\begin{aligned} x&=4 \\ y&=5 \end{aligned}\right\};\ \text{or}\ \left.\begin{aligned} x&=5 \\ y&=4 \end{aligned}\right\}.$$

3. Solve

$$\left.\begin{aligned} x - y &= 4 \qquad (1)\\ x^2 + y^2 &= 40 \qquad (2)\end{aligned}\right\}$$

Square (1), $x^2 - 2xy + y^2 = 16.$ (3)

Subtract (2) from (3), $-2xy = -24.$ (4)

Subtract (4) from (2),

$$x^2 + 2xy + y^2 = 64.$$

Extract the root, $x + y = \pm 8.$ (5)

From (1) and (5),

$$\left.\begin{aligned} x &= 6\\ y &= 2\end{aligned}\right\}; \text{ or } \left.\begin{aligned} x &= -2\\ y &= -6\end{aligned}\right\}.$$

4. Solve

$$\left.\begin{aligned} x^3 + y^3 &= 91 \qquad (1)\\ x + y &= 7 \qquad (2)\end{aligned}\right\}$$

Divide (1) by (2), $x^2 - xy + y^2 = 13.$ (3)

Square (2), $x^2 + 2xy + y^2 = 49.$ (4)

Subtract (3) from (4), $3xy = 36.$

Divide by -3, $-xy = -12.$ (5)

Add (5) and (3), $x^2 - 2xy + y^2 = 1.$

Extract the root, $x - y = \pm 1.$ (6)

From (2) and (6),

$$\left.\begin{aligned} x &= 4\\ y &= 3\end{aligned}\right\}; \text{ or } \left.\begin{aligned} x &= 3\\ y &= 4\end{aligned}\right\}.$$

5. Solve

$$\left.\begin{aligned} x^3 + y^3 &= 18xy \qquad (1)\\ x + y &= 12 \qquad (2)\end{aligned}\right\}$$

Divide (1) by (2), $x^2 - xy + y^2 = \dfrac{3xy}{2}.$ (3)

Square (2), $x^2 + 2xy + y^2 = 144.$ (4)

Subtract (4) from (3), $-3xy = \dfrac{3xy}{2} - 144,$

which gives $xy = 32.$

We now have,

$$\left.\begin{aligned} x + y &= 12\\ xy &= 32\end{aligned}\right\}.$$

Solving, we find,

$$\left.\begin{aligned} x &= 8\\ y &= 4\end{aligned}\right\}; \text{ or } \left.\begin{aligned} x &= 4\\ y &= 8\end{aligned}\right\}.$$

EXERCISE 121.

Solve:

1. $\left.\begin{aligned} x+y&=7\\ xy&=10 \end{aligned}\right\}$

2. $\left.\begin{aligned} x+y&=12\\ xy&=27 \end{aligned}\right\}$

3. $\left.\begin{aligned} x-y&=6\\ xy&=-8 \end{aligned}\right\}$

4. $\left.\begin{aligned} x-y&=10\\ xy&=11 \end{aligned}\right\}$

5. $\left.\begin{aligned} x+y&=12\\ x^2+y^2&=80 \end{aligned}\right\}$

6. $\left.\begin{aligned} x+y&=3\\ x^2+y^2&=29 \end{aligned}\right\}$

7. $\left.\begin{aligned} x-y&=9\\ x^2+y^2&=45 \end{aligned}\right\}$

8. $\left.\begin{aligned} x+2y&=7\\ x^2+y^2&=10 \end{aligned}\right\}$

9. $\left.\begin{aligned} 3x-y&=12\\ x^2-y^2&=16 \end{aligned}\right\}$

10. $\left.\begin{aligned} y&=3x+1\\ x^2+xy&=33 \end{aligned}\right\}$

11. $\left.\begin{aligned} 5x-4y&=10\\ 3x^2-4y^2&=8 \end{aligned}\right\}$

12. $\left.\begin{aligned} x+7y&=23\\ xy&=6 \end{aligned}\right\}$

13. $\left.\begin{aligned} 2x-3y&=2\\ x^2-2xy&=-7 \end{aligned}\right\}$

14. $\left.\begin{aligned} 2x-3y&=1\\ 3x^2-4xy&=32 \end{aligned}\right\}$

15. $\left.\begin{aligned} x^2-xy+y^2&=21\\ x+y&=9 \end{aligned}\right\}$

16. $\left.\begin{aligned} x^2-3xy+2y^2&=0\\ 2x+3y&=7 \end{aligned}\right\}$

17. $\left.\begin{aligned} x^2-y^2&=9\\ x-y&=1 \end{aligned}\right\}$

18. $\left.\begin{aligned} x^2+3y+17&=0\\ 3x-y&=3 \end{aligned}\right\}$

19. $\left.\begin{aligned} \frac{1}{x}+\frac{1}{y}&=5\\ \frac{1}{x^2}-\frac{1}{y^2}&=5 \end{aligned}\right\}$

20. $\left.\begin{aligned} \frac{1}{x}+\frac{1}{y}&=\frac{5}{6}\\ \frac{1}{x^2}+\frac{1}{y^2}&=\frac{13}{36} \end{aligned}\right\}$

21. $\left.\begin{aligned} 3x+2y&=2xy\\ xy&=6 \end{aligned}\right\}$

22. $\left.\begin{aligned} \frac{1}{x}+\frac{1}{y}&=11\\ \frac{1}{x^2}+\frac{1}{y^2}&=61 \end{aligned}\right\}$

23. $\left.\begin{aligned} 8x+6y&=4xy\\ xy&=16 \end{aligned}\right\}$

24. $\left.\begin{aligned} x^3+y^3&=35\\ x+y&=5 \end{aligned}\right\}$

25. $\left.\begin{aligned} x^3 - y^3 &= 61 \\ x - y &= 1 \end{aligned}\right\}$

26. $\left.\begin{aligned} x^3 + y^3 &= 65 \\ x + y &= 5 \end{aligned}\right\}$

27. $\left.\begin{aligned} x^2y + xy^2 &= 120 \\ x + y &= 8 \end{aligned}\right\}$

28. $\left.\begin{aligned} x^3 - y^3 &= \tfrac{7}{64} \\ x - y &= \tfrac{1}{4} \end{aligned}\right\}$

29. $\left.\begin{aligned} x^3 + y^3 &= 126 \\ x^2 - xy + y^2 &= 21 \end{aligned}\right\}$

30. $\left.\begin{aligned} x^3 - y^3 &= 56 \\ x^2 + xy + y^2 &= 28 \end{aligned}\right\}$

31. $\left.\begin{aligned} \frac{x^2}{y} + \frac{y^2}{x} &= \frac{35}{3} \\ \frac{1}{x} + \frac{1}{y} &= \frac{5}{12} \end{aligned}\right\}$

32. $\left.\begin{aligned} \frac{x^2}{y} - \frac{y^2}{x} &= \frac{19}{2} \\ \frac{1}{y} - \frac{1}{x} &= \frac{1}{18} \end{aligned}\right\}$

33. $\left.\begin{aligned} x^2 + xy &= 24 \\ xy + y^2 &= 40 \end{aligned}\right\}$

34. $\left.\begin{aligned} x^2 - xy &= 8 \\ xy - y^2 &= 7 \end{aligned}\right\}$

35. $\left.\begin{aligned} x^2 + 2xy &= 24 \\ 2xy + 4y^2 &= 120 \end{aligned}\right\}$

36. $\left.\begin{aligned} 4x^2 + 5xy &= 14 \\ 7xy + 9y^2 &= 50 \end{aligned}\right\}$

37. $\left.\begin{aligned} x^2 + xy + y^2 &= 39 \\ 2x^2 + 3xy + y^2 &= 63 \end{aligned}\right\}$

38. $\left.\begin{aligned} x^2 + 3y^2 &= 52 \\ xy + 2y^2 &= 40 \end{aligned}\right\}$

39. $\left.\begin{aligned} 2x^2 - y^2 &= 46 \\ xy + y^2 &= 14 \end{aligned}\right\}$

40. $\left.\begin{aligned} x^2 + xy + 2y^2 &= 44 \\ 2x^2 - 3xy + 2y^2 &= 16 \end{aligned}\right\}$

41. $\left.\begin{aligned} x^2 + 3y^2 &= 31 \\ 4xy + y^2 &= 33 \end{aligned}\right\}$

42. $\left.\begin{aligned} 3x^2 + 7xy &= 82 \\ x^2 + 5xy + 9y^2 &= 279 \end{aligned}\right\}$

43. $\left.\begin{aligned} x^4 + y^4 &= 97 \\ x + y &= 5 \end{aligned}\right\}$

44. $\left.\begin{aligned} x^4 + y^4 &= 17 \\ x + y &= 3 \end{aligned}\right\}$

45. $\left.\begin{aligned} x^4 + y^4 &= 881 \\ x - y &= 1 \end{aligned}\right\}$

46. $\left.\begin{aligned} x^5 + y^5 &= 211 \\ x + y &= 1 \end{aligned}\right\}$

47. $\left.\begin{aligned} x^5 - y^5 &= 242 \\ x - y &= 2 \end{aligned}\right\}$

48. $\left.\begin{aligned} x^2 + y^2 &= xy + 7 \\ x + y &= xy - 1 \end{aligned}\right\}$

49. $\left.\begin{aligned} x^3 - y^3 &= 7xy \\ x - y &= 2 \end{aligned}\right\}$

50. $$\left.\begin{array}{l} x^3 + y^3 = 36\,xy \\ x + y = 24 \end{array}\right\}$$

51. $$\left.\begin{array}{l} x^3 + 3\,xy^2 = 62 \\ 3\,x^2y + y^3 = 63 \end{array}\right\}$$

52. $$\left.\begin{array}{l} x^2 + xy + y^2 = 61 \\ x^4 + x^2y^2 + y^4 = 1281 \end{array}\right\}$$

53. $$\left.\begin{array}{l} x^2 - xy + y^2 = 3 \\ x^4 + x^2y^2 + y^4 = 21 \end{array}\right\}$$

54. $$\left.\begin{array}{l} \dfrac{x+y}{x-y} + \dfrac{x-y}{x+y} = \dfrac{10}{3} \\ x^2 + y^2 = 20 \end{array}\right\}$$

55. $$\left.\begin{array}{l} \dfrac{x-y}{x+y} - \dfrac{x+y}{x-y} = \dfrac{24}{5} \\ 3\,x + 4\,y = 36 \end{array}\right\}$$

56. $$\left.\begin{array}{l} x^2 + y^2 + x + y = 32 \\ xy + 16 = 0 \end{array}\right\}$$

57. $$\left.\begin{array}{l} x - y - 3 = 0 \\ 2\,(x^2 - y^2) = 3\,xy \end{array}\right\}$$

58. $$\left.\begin{array}{l} \dfrac{1}{x} + \dfrac{1}{y} = 7 \\ \dfrac{1}{x+1} + \dfrac{1}{y+1} = \dfrac{31}{20} \end{array}\right\}$$

59. $$\left.\begin{array}{l} x^4 + y^4 = 272 \\ x^2 + y^2 = 3\,xy - 4 \end{array}\right\}$$

60. $$\left.\begin{array}{l} x^2 + y^2 = x^2y^2 + 1 \\ x + y = 2\,xy - 1 \end{array}\right\}$$

61. $$\left.\begin{array}{l} x^3 - y^3 = a^3 \\ x - y = a \end{array}\right\}$$

62. $$\left.\begin{array}{l} \dfrac{x}{a} + \dfrac{y}{b} = 1 \\ \dfrac{a}{x} + \dfrac{b}{y} = 4 \end{array}\right\}$$

63. $$\left.\begin{array}{l} x^2 = ax + by \\ y^2 = bx + ay \end{array}\right\}$$

64. $$\left.\begin{array}{l} x^2 + y^2 = 2\,(a^2 + b^2) \\ xy = a^2 - b^2 \end{array}\right\}$$

65. $$\left.\begin{array}{l} x^2 + y^2 = \dfrac{a^4 + b^4}{a^2b^2} \\ xy = 1 \end{array}\right\}$$

66. $$\left.\begin{array}{l} x^2 - y^2 = \dfrac{a-b}{a+b} \\ xy = \dfrac{ab}{(a+b)^2} \end{array}\right\}$$

67. $$\left.\begin{array}{l} x^2 - xy = 2\,ab + 2\,b^2 \\ xy - y^2 = 2\,ab - 2\,b^2 \end{array}\right\}$$

68. $$\left.\begin{array}{l} x^2 - y^2 = a^2 \\ xy = b^2 \end{array}\right\}$$

69. $$\left.\begin{array}{l} x^2 - y^2 = 8\,ab \\ xy = a^2 - 4\,b^2 \end{array}\right\}$$

70. $$\left.\begin{array}{l} x^3 + y^3 = a^3 + b^3 \\ x + y = a + b \end{array}\right\}$$

EXERCISE 122.

1. The area of a rectangle is 60 square feet, and its perimeter is 34 feet. Find the length and breadth of the rectangle.

2. The area of a rectangle is 108 square feet. If the length and breadth of a rectangle are each increased by 3 feet, the area will be 180 square feet. Find the length and breadth of the rectangle.

3. If the length and breadth of a rectangular plot are each increased by 10 feet, the area will be increased by 400 square feet. But if the length and breadth are each diminished by 5 feet, the area will be 75 square feet. Find the length and breadth of the plot.

4. The area of a rectangle is 168 square feet, and the length of its diagonal is 25 feet. Find the length and breadth of the rectangle.

5. The diagonal of a rectangle is 25 inches. If the rectangle were 4 inches shorter and 8 inches wider, the diagonal would still be 25 inches. Find the area of the rectangle.

6. A rectangular field, containing 180 square rods, is surrounded by a road 1 rod wide. The area of the road is 58 square rods. Find the dimensions of the field.

7. Two square gardens have a total surface of 2137 square yards. A rectangular piece of land whose dimensions are respectively equal to the sides of the two squares will have 1093 square yards less than the two gardens united. What are the sides of the two squares?

8. The sum of two numbers is 22, and the difference of their squares is 44. Find the numbers.

9. The difference of two numbers is 6, and their product exceeds their sum by 39. Find the numbers.

10. The sum of two numbers is equal to the difference of their squares, and the product of the numbers exceeds twice their sum by 2. Find the numbers.

11. The sum of two numbers is 20, and the sum of their cubes is 2060. Find the numbers.

12. The difference of two numbers is 5, and the difference of their cubes exceeds the difference of their squares by 1290. Find the numbers.

13. A number is formed of two digits. The sum of the squares of the digits is 58. If twelve times the units' digit is subtracted from the number, the order of the digits will be reversed. Find the number.

14. A number is formed of three digits, the third digit being twice the sum of the other two. The first digit plus the product of the other two digits is 25. If 180 is added to the number, the order of the first and second digits will be reversed. Find the number.

15. There are two numbers formed of the same two digits in reverse order. The sum of the numbers is 33 times the difference of the two digits, and the difference of the squares of the numbers is 4752. Find the numbers.

16. The sum of the numerator and denominator of a certain fraction is 5; and if the numerator and denominator are each increased by 3, the value of the fraction will be increased by $\frac{1}{6}$. Find the fraction.

17. The fore wheel of a carriage turns in a mile 132 times more than the hind wheel; but if the circumferences were each increased by 2 feet, it would turn only 88 times more. Find the circumference of each.

CHAPTER XXI.

RATIO, PROPORTION, AND VARIATION.

316. The *relative magnitude* of two numbers is called their **ratio,** when expressed by the fraction which the first is of the second.

Thus, the ratio of 6 to 3 is indicated by the fraction $\frac{6}{3}$, which is sometimes written 6 : 3.

317. The first term of a ratio is called the **antecedent,** and the second term the **consequent.** When the antecedent is *equal* to the consequent, the ratio is called a *ratio of equality;* when the antecedent is *greater* than the consequent, the ratio is called a *ratio of greater inequality;* when *less,* a *ratio of less inequality.*

318. When the antecedent and consequent are interchanged, the resulting ratio is called the *inverse* of the given ratio.

Thus, the ratio 3 : 6 is the *inverse* of the ratio 6 : 3.

319. The ratio of two *quantities* that can be expressed in *integers* in terms of a *common unit* is equal to the ratio of the two *numbers* by which they are expressed.

Thus, the ratio of $9 to $11 is equal to the ratio of 9 : 11; and the ratio of a line $2\frac{2}{3}$ inches long to a line $3\frac{3}{4}$ inches long, when both are expressed in terms of a unit $\frac{1}{12}$ of an inch long, is equal to the ratio of 32 : 45.

320. Two quantities *different in kind* can have no ratio, for then one cannot be a fraction of the other.

321. Two quantities that can be expressed in integers in terms of a common unit are said to be *commensurable*. The common unit is called a *common measure*, and each quantity is called a *multiple* of this common measure.

Thus, a common measure of $2\frac{1}{2}$ feet and $3\frac{2}{3}$ feet is $\frac{1}{6}$ of a foot, which is contained 15 times in $2\frac{1}{2}$ feet, and 22 times in $3\frac{2}{3}$ feet. Hence, $2\frac{1}{2}$ feet and $3\frac{2}{3}$ feet are multiples of $\frac{1}{6}$ of a foot, $2\frac{1}{2}$ feet being obtained by taking $\frac{1}{6}$ of a foot 15 times, and $3\frac{2}{3}$ feet by taking $\frac{1}{6}$ of a foot 22 times.

322. When two quantities are *incommensurable*, that is, have no common unit in terms of which *both* quantities can be expressed in *integers*, it is impossible to find a fraction that will indicate the **exact value** of the ratio of the given quantities. It is possible, however, by taking the unit sufficiently small, to find a fraction that shall differ from the true value of the ratio by as little as we please.

Thus, if a and b denote the diagonal and side of a square,

$$\frac{a}{b} = \sqrt{2}.$$

Now $\sqrt{2} = 1.41421356\cdots$, a value greater than 1.414213, but less than 1.414214.

If, then, a *millionth part* of b is taken as the unit, the value of the ratio $\frac{a}{b}$ lies between $\frac{1414213}{1000000}$ and $\frac{1414214}{1000000}$, and therefore differs from either of these fractions by less than $\frac{1}{1000000}$.

By carrying the decimal further, a fraction may be found that will differ from the true value of the ratio by less than a *billionth*, *trillionth*, *or any other assigned value whatever.*

323. Expressed generally, when a and b are incommensurable, and b is divided into any integral number (n) of equal parts, if one of these parts is contained in a more than m times, but less than $m + 1$ times, then

$$\frac{a}{b} > \frac{m}{n}, \text{ but } < \frac{m+1}{n};$$

that is, the value of $\frac{a}{b}$ *lies between* $\frac{m}{n}$ and $\frac{m+1}{n}$.

The error, therefore, in taking either of these values for $\frac{a}{b}$ is less than the difference between $\frac{m}{n}$ and $\frac{m+1}{n}$; that is, less than $\frac{1}{n}$. But by increasing n indefinitely, $\frac{1}{n}$ can be made to decrease indefinitely, and to become less than any assigned value, however small, though it cannot be made absolutely equal to zero.

324. The ratio between two incommensurable quantities is called an **incommensurable ratio.**

325. THEOREM. *Two incommensurable ratios are equal if, when the unit of measure is indefinitely diminished, their approximate values constantly remain equal.*

Let $a:b$ and $a':b'$ be two incommensurable ratios whose true values lie *between* the approximate values $\frac{m}{n}$ and $\frac{m+1}{n}$, when the unit of measure is indefinitely diminished. Then they cannot *differ by so much as* $\frac{1}{n}$.

Now the difference (if any) between the *fixed values* $a:b$ and $a':b'$ is a *fixed value.* Let d denote this fixed value.

Then, $$d < \frac{1}{n}.$$

But if d has *any* value, *however small,* $\frac{1}{n}$, which by hypothesis can be made less than *any* value, however small, can be made less than d.

Therefore, d cannot have *any* value; that is, $d = 0$, and there is no difference between the ratios $a:b$ and $a':b'$; therefore, $a:b = a':b'$.

326. *A ratio will not be altered if both its terms are multiplied by the same positive number.*

For the ratio $a : b$ is represented by $\frac{a}{b}$, the ratio $ma : mb$ is represented by $\frac{ma}{mb}$; and since $\frac{ma}{mb} = \frac{a}{b}$, therefore, $ma : mb = a : b$.

327. *A ratio will be altered if its terms are multiplied by different positive numbers; and will be increased or diminished according as the multiplier of the antecedent is greater than or less than that of the consequent.*

For $ma : nb > \text{or} < a : b$

according as $\frac{ma}{nb} > \text{or} < \frac{a}{b}\left(= \frac{na}{nb}\right)$,

according as $ma > \text{or} < na$,

according as $m > \text{or} < n$.

328. *A ratio of greater inequality will be diminished, and a ratio of less inequality increased, by adding the same positive number to both its terms.*

For $a + x : b + x > \text{or} < a : b$,

according as $\frac{a+x}{b+x} > \text{or} < \frac{a}{b}$,

according as $ab + bx > \text{or} < ab + ax$,

according as $bx > \text{or} < ax$,

according as $b > \text{or} < a$.

329. *A ratio of greater inequality will be increased, and a ratio of less inequality diminished, by subtracting the same positive number from both its terms.*

For $a - x : b - x > \text{or} < a : b$,

according as $\frac{a-x}{b-x} > \text{or} < \frac{a}{b}$,

according as $ab - bx > \text{or} < ab - ax$,

according as $ax > \text{or} < bx$,

according as $a > \text{or} < b$.

330. Ratios are *compounded* by taking the product of the fractions that represent them.

Thus, the ratio compounded of $a : b$ and $c : d$ is found by taking the product of $\frac{a}{b}$ and $\frac{c}{d} = \frac{ac}{bd}$.

The ratio compounded of $a : b$ and $a : b$ is the *duplicate* ratio $a^2 : b^2$, and the ratio compounded of $a : b$, $a : b$, and $a : b$ is the *triplicate* ratio $a^3 : b^3$.

331. Ratios are *compared* by comparing the fractions that represent them.

Thus, $a : b >$ or $< c : d$

according as $\frac{a}{b} >$ or $< \frac{c}{d}$,

according as $\frac{ad}{bd} >$ or $< \frac{bc}{bd}$,

according as $ad >$ or $< bc$.

Exercise 123.

1. Write the ratio compounded of $3 : 5$ and $8 : 7$. Which of these ratios is increased, and which is diminished by the composition?

2. Compound the duplicate ratio of $4 : 15$ with the triplicate of $5 : 2$.

3. Show that a duplicate ratio is greater or less than its simple ratio according as it is a ratio of greater or less inequality.

4. Arrange in order of magnitude the ratios $3 : 4$; $23 : 25$; $10 : 11$; and $15 : 16$.

5. Arrange in order of magnitude

$$a + b : a - b \text{ and } a^2 + b^2 : a^2 - b^2, \text{ if } a > b.$$

Find the ratio compounded of:

6. $3 : 5$; $10 : 21$; $14 : 15$. 7. $7 : 9$; $102 : 105$; $15 : 17$.

8. $\dfrac{a^2 + ax + x^2}{a^3 - a^2x + ax^2 - x^3}$ and $\dfrac{a^2 - ax + x^2}{a + x}$.

9. $\dfrac{x^2 - 9x + 20}{x^2 - 6x}$ and $\dfrac{x^2 - 13x + 42}{x^2 - 5x}$.

10. $a + b : a - b$; $a^2 + b^2 : (a + b)^2$; $(a^2 - b^2)^2 : a^4 - b^4$.

11. Two numbers are in the ratio 2 : 3, and if 9 is added to each, they are in the ratio 3 : 4. Find the numbers.

Let $2x$ and $3x$ represent the numbers.

12. Show that the ratio $a : b$ is the duplicate of the ratio $a + c : b + c$, if $c^2 = ab$.

13. Two numbers are in the ratio 3 : 4. Their sum is to the sum of their squares as 7 : 50. Find the numbers.

14. If five gold and four silver coins are worth as much as three gold and twelve silver coins, find the ratio of the value of a gold coin to that of a silver coin.

15. If eight gold and nine silver coins are worth as much as six gold and nineteen silver coins, find the ratio of the value of a silver coin to that of a gold coin.

16. There are two roads from A to B, one of them 14 miles longer than the other; and two roads from B to C, one of them 8 miles longer than the other. The distance from A to B is to the distance from B to C, by the shorter roads, as 1 to 2; by the longer roads, as 2 to 3. Find the distances.

17. What must be added to each of the terms of the ratio $m : n$, that it may become equal to the ratio $p : q$?

18. A rectangular field contains 5270 acres, and its length is to its breadth in the ratio of 31 : 17. Find its dimensions.

Proportion.

332. An equation consisting of two equal ratios is called a **proportion;** and the terms of the ratios are called **proportionals.**

333. The algebraic test of a proportion is that the two fractions which represent the ratios of the quantities compared shall be equal.

Thus, the ratio $a:b$ is equal to the ratio $c:d$ if the fraction that represents the ratio $a:b$ is equal to the fraction that represents the ratio $c:d$. Then the four quantities, a, b, c, d, are called **proportionals,** or are said to be in **proportion.**

334. If the ratios $a:b$ and $c:d$ form a proportion, the proportion is written

$$a:b=c:d$$

(read the ratio of a to b is equal to the ratio of c to d),

or
$$a:b::c:d$$

(read a is to b in the same ratio as c is to d).

The first and last terms, a and d, are called the **extremes.**

The two middle terms, b and c, are called the **means.**

335. In the proportion $a:b=c:d$; d is called a *fourth proportional* to a, b, and c.

In the proportion $a:b=b:c$; c is called a *third proportional* to a and b.

In the proportion $a:b=b:c$; b is called a *mean proportional* between a and c.

336. A *continued proportion* is a series of equal ratios in which each consequent is the same as the next antecedent.

Thus, $a:b=b:c=c:d=d:e=e:f$ is a continued proportion.

337. *When four quantities are in proportion, the product of the extremes is equal to the product of the means.*

For, if $a : b = c : d,$

then, $\frac{a}{b} = \frac{c}{d}.$

Multiply by bd, $ad = bc.$

The equation $ad = bc$ gives $a = \frac{bc}{d}$, $b = \frac{ad}{c}$; so that an extreme may be found by dividing the product of the means by the other extreme; and a mean may be found by dividing the product of the extremes by the other mean.

Note. By the *product of two quantities* we mean the product of the two numbers that represent them when the quantities are expressed in a common unit.

338. *If the product of two quantities is equal to the product of two others, either two may be made the extremes of a proportion and the other two the means.*

For, if $ad = bc,$

then, divide by bd, $\frac{ad}{bd} = \frac{bc}{bd},$

or $\frac{a}{b} = \frac{c}{d}.$

$$\therefore a : b = c : d.$$

339. **Transformations of a Proportion.** *If four quantities, a, b, c, d, are in proportion, they will be in proportion by:*

I. **Inversion**; that is, b will be to a as d is to c.

For, if $a : b = c : d,$

then, $\frac{a}{b} = \frac{c}{d},$

and the reciprocals of these fractions are equal;

that is, $$\frac{b}{a}=\frac{d}{c}.$$

$$\therefore b:a=d:c.$$

II. **Composition**; that is, $a+b$ will be to b as $c+d$ is to d.

For, if $$a:b=c:d,$$

then, $$\frac{a}{b}=\frac{c}{d},$$

and $$\frac{a}{b}+1=\frac{c}{d}+1,$$

or $$\frac{a+b}{b}=\frac{c+d}{d}.$$

$$\therefore a+b:b=c+d:d.$$

III. **Division**; that is, $a-b$ will be to b as $c-d$ is to d.

For, if $$a:b=c:d,$$

then, $$\frac{a}{b}=\frac{c}{d},$$

and $$\frac{a}{b}-1=\frac{c}{d}-1,$$

or $$\frac{a-b}{b}=\frac{c-d}{d}.$$

$$\therefore a-b:b=c-d:d.$$

IV. **Composition and Division**; that is, $a+b$ will be to $a-b$ as $c+d$ is to $c-d$.

For, from II, $$\frac{a+b}{b}=\frac{c+d}{d},$$

and from III, $$\frac{a-b}{b}=\frac{c-d}{d}.$$

Divide, $$\frac{a+b}{a-b}=\frac{c+d}{c-d}.$$

$$\therefore a+b:a-b=c+d:c-d.$$

V. **Alternation**; that is, a will be to c as b is to d.

For, if $$a : b = c : d,$$

then, $$\frac{a}{b} = \frac{c}{d}.$$

Multiply by $\frac{b}{c}$, $$\frac{ab}{bc} = \frac{bc}{cd},$$

or $$\frac{a}{c} = \frac{b}{d}.$$

$$\therefore a : c = b : d.$$

NOTE. In order for four quantities, a, b, c, d, to be in proportion, a and b must be of the *same kind* and c and d of the same kind; but c and d need not necessarily be of the same kind as a and b. In applying *alternation*, however, *all four quantities must be of the same kind.*

340. In a Series of Equal Ratios, *the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.*

For, if $$\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{g}{h},$$

r may be put for each of these ratios.

Then, $$\frac{a}{b} = r,\ \frac{c}{d} = r,\ \frac{e}{f} = r,\ \frac{g}{h} = r.$$

$$\therefore a = br,\ c = dr,\ e = fr,\ g = hr.$$

$$\therefore a + c + e + g = (b + d + f + h)\,r.$$

$$\therefore \frac{a + c + e + g}{b + d + f + h} = r = \frac{a}{b}.$$

$$\therefore a + c + e + g : b + d + f + h = a : b.$$

In like manner it may be shown that

$$ma + nc + pe + qg : mb + nd + pf + qh = a : b.$$

341. *A mean proportional between two quantities is equal to the square root of their product.*

For, if $$a : b = b : c,$$

then, $$\frac{a}{b} = \frac{b}{c}.$$

Clear of fractions, $$b^2 = ac.$$

Extract the square root, $$b = \sqrt{ac}.$$

342. *The products of the corresponding terms of two or more proportions are in proportion.*

For, if $$a : b = c : d,$$ $$e : f = g : h,$$ $$k : l = m : n,$$

then, $$\frac{a}{b} = \frac{c}{d}, \ \frac{e}{f} = \frac{g}{h}, \ \frac{k}{l} = \frac{m}{n}.$$

Take the product of the left members, and also of the right members of these equations,

$$\frac{aek}{bfl} = \frac{cgm}{dhn}.$$

$$\therefore aek : bfl = cgm : dhn.$$

343. *Like powers, or like roots, of the terms of a proportion are in proportion.*

For, if $$a : b = c : d,$$

then, $$\frac{a}{b} = \frac{c}{d}.$$

Raise both sides to the *n*th power,

$$\frac{a^n}{b^n} = \frac{c^n}{d^n}.$$

$$\therefore a^n : b^n = c^n : d^n.$$

Extract the nth root, $$\frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}}=\frac{c^{\frac{1}{n}}}{d^{\frac{1}{n}}}.$$

$$\therefore a^{\frac{1}{n}} : b^{\frac{1}{n}} = c^{\frac{1}{n}} : d^{\frac{1}{n}}.$$

344. The laws that have been established for ratios should be remembered when ratios are expressed in fractional form.

1. Solve $\dfrac{x^2+x+1}{x^2-x-1}=\dfrac{x^2-x+2}{x^2+x-2}$.

By composition and division,

$$\frac{2x^2}{2(x+1)}=\frac{2x^2}{-2(x-2)}.$$

This equation is satisfied when $x=0$. For any other value of x, we may divide by x^2.

We then have $$\frac{1}{x+1}=\frac{1}{2-x},$$

and therefore, $$x=\tfrac{1}{2}.$$

2. If $a:b=c:d$, show that

$$a^2+ab:b^2-ab=c^2+cd:d^2-cd.$$

If $$\frac{a}{b}=\frac{c}{d},$$

then, $$\frac{a+b}{a-b}=\frac{c+d}{c-d}, \qquad (\S\ 339,\ \text{IV})$$

and $$\frac{a}{-b}=\frac{c}{-d}.$$

$$\therefore \frac{a}{-b}\times\frac{a+b}{a-b}=\frac{c}{-d}\times\frac{c+d}{c-d}; \qquad (\S\ 342)$$

that is, $$\frac{a^2+ab}{b^2-ab}=\frac{c^2+cd}{d^2-cd},$$

or $$a^2+ab:b^2-ab=c^2+cd:d^2-cd.$$

EXERCISE 124.

1. Find a third proportional to 21 and 28.

2. Find a mean proportional between 6 and 24.

3. Find a fourth proportional to 3, 5, and 42.

4. Find x if $5 + x : 11 - x = 3 : 5$.

If $a : b = c : d$, show that:

5. $ac : bd = c^2 : d^2$.

6. $ab : cd = a^2 : c^2$.

7. $a^2 - b^2 : c^2 - d^2 = a^2 : c^2$.

8. $2a + b : 2c + d = b : d$.

9. $5a - b : 5c - d = a : c$.

10. $a - 3b : a + 3b = c - 3d : c + 3d$.

11. $a^2 + ab + b^2 : a^2 - ab + b^2 = c^2 + cd + d^2 : c^2 - cd + d^2$.

Find x in the proportion:

12. $45 : 68 = 90 : x$.

13. $6 : 3 = x : 7$.

14. $x : 1\frac{1}{2} = 1\frac{5}{7} : 1\frac{4}{5}$.

15. $3 : x = 7 : 42$.

16. Find two numbers in the ratio $2 : 3$, the sum of whose squares is 325.

17. Find two numbers in the ratio $5 : 3$, the difference of whose squares is 400.

18. Find three numbers which are to each other as $2 : 3 : 5$, such that half the sum of the greatest and least exceeds the other by 25.

19. A and B trade with different sums. A gains \$200 and B loses \$50 and now A's stock : B's : : $2 : \frac{1}{2}$. But, if A had gained \$100 and B lost \$85, their stocks would have been as $15 : 3\frac{1}{4}$. Find the original stock of each.

20. Find x if $6x - a : 4x - b = 3x + b : 2x + a$.

21. Find x and y from the proportions

$$x : y = x + y : 42; \quad x : y = x - y : 6$$

22. Find x and y from the proportions

$$2x + y : y = 3y : 2y - x;$$
$$2x + 1 : 2x + 6 = y : y + 2.$$

23. If $\dfrac{a+b+c+d}{a+b-c-d} = \dfrac{a-b+c-d}{a-b-c+d}$, show that $\dfrac{a}{b} = \dfrac{c}{d}$.

Variation.

345. One quantity is said to *vary as* another, when the two quantities are so related that the ratio of any two values of the one is equal to the ratio of the corresponding values of the other.

Thus, if it is said that the weight of water varies as its volume, the meaning is, that *one* gallon of water is to *any number* of gallons as the weight of *one* gallon is to the weight of the *given number* of gallons.

346. Two quantities may be so related that when a value of one is given, the corresponding value of the other can be found. In this case one quantity is said to be a *function* of the other; that is, one quantity depends upon the other for its value. Thus, if the rate at which a man walks is known, the distance he walks can be found when the time is given; the distance is in this case a *function* of the time.

347. There is an unlimited number of ways in which two quantities may be related. We shall consider in this chapter only a few of these ways.

348. When x and y are so related that their ratio is *constant*, y is said to vary as x; this is abbreviated thus:

$y \propto x$. The sign $\propto$, called the **sign of variation,** is read *varies as.*

Thus, the area of a triangle with a given base varies as its altitude; for, if the altitude is changed in any ratio, the area will be changed in the same ratio.

In this case, if we represent the constant ratio by m,

$$y : x = m, \text{ or } \frac{y}{x} = m; \therefore y = mx.$$

Again, if y', x' and y'', x'' are two sets of corresponding values of y and x,

then, $$y' : x' = y'' : x'',$$
or $$y' : y'' = x' : x''. \qquad (\S\ 339, \text{V})$$

349. When x and y are so related that the ratio of y to $\frac{1}{x}$ is constant, y is said to vary *inversely* as x; this is written $y \propto \frac{1}{x}$.

Thus, the time required to do a certain amount of work varies inversely as the number of workmen employed; for, if the number of workmen is doubled, halved, or changed in any other ratio, the time required will be halved, doubled, or changed in the inverse ratio.

In this case, $y : \frac{1}{x} = m$; $\therefore y = \frac{m}{x}$, and $xy = m$; that is, the product xy is constant.

As before, $$y' : \frac{1}{x'} = y'' : \frac{1}{x''},$$
$$x'y' = x''y'',$$
or $$y' : y'' = x'' : x'. \qquad (\S\ 338)$$

350. If the ratio of $y : xz$ is constant, then y is said to *vary jointly* as x and z.

In this case, $$y = mxz,$$
and $$y' : y'' = x'z' : x''z''.$$

351. If the ratio $y : \frac{x}{z}$ is constant, then y varies *directly* as x and *inversely* as z.

In this case, $$y = \frac{mx}{z},$$

and $$y' : y'' = \frac{mx'}{z'} : \frac{mx''}{z''} = \frac{x'}{z'} : \frac{x''}{z''}.$$

352. THEOREM 1. *If $y \infty x$, and $x \infty z$, then $y \infty z$.*

For $$y = mx \text{ and } x = nz.$$
$$\therefore y = mnz;$$

since mn is constant, y varies as z.

353. THEOREM 2. *If $y \infty x$, and $z \infty x$, then $(y \pm z) \infty x$.*

For $$y = mx \text{ and } z = nx.$$
$$\therefore y \pm z = (m \pm n)\, x;$$

since $m \pm n$ is constant, $y \pm z$ varies as x.

354. THEOREM 3. *If $y \propto x$ when z is constant, and $y \propto z$ when x is constant, then $y \infty xz$ when x and z are both variable.*

Let x', y', z', and x'', y'', z'' be corresponding values of the variables.

Let x change from x' to x'', z remaining constant, and let the corresponding value of y be Y.

Then, $$y' : Y = x' : x''. \tag{1}$$

Now let z change from z' to z'', x remaining constant.

Then, $$Y : y'' = z' : z''. \tag{2}$$

Multiply (1) and (2),

$$y'Y : y''Y = x'z' : x''z'', \qquad (\S\ 342)$$

or $$y' : y'' = x'z' : x''z'',$$

or $$y' : x'z' = y'' : x''z''. \qquad (\S\ 339,\ \text{V})$$

$\therefore$ the ratio $y : xz$ is constant, and y varies as xz.

In like manner it may be shown that if y varies as each of any number of quantities x, z, u, etc., when the rest are unchanged, then when they all change, $y \propto xzu$, etc.

Thus, the volume of a rectangular solid varies as the length when the width and thickness remain constant; as the width when the length and thickness remain constant; as the thickness when the length and width remain constant; but as the product of length, breadth, and thickness when all three vary.

1. If y varies inversely as x, and when $y = 2$ the corresponding value of x is 36, find the corresponding value of x when $y = 9$.

Here $$y = \frac{m}{x}, \text{ or } m = xy.$$
$$\therefore m = 2 \times 36 = 72.$$

If 9 and 72 are substituted for y and m, respectively,

the result is $$9 = \frac{72}{x}, \text{ or } 9x = 72.$$
$$\therefore x = 8.$$

2. The weight of a sphere of given material varies as its volume, and its volume varies as the cube of its diameter. If a sphere 4 inches in diameter weighs 20 pounds, find the weight of a sphere 5 inches in diameter.

Let W represent the weight,
V represent the volume,
and D represent the diameter.

Then, $$W \propto V \text{ and } V \propto D^3.$$
$$\therefore W \propto D^3. \qquad (\S\ 352)$$

Put $$W = mD^3;$$

then, since 20 and 4 are corresponding values of W and D.
$$20 = m \times 64.$$
$$\therefore m = \frac{20}{64} = \frac{5}{16}.$$
$$\therefore W = \tfrac{5}{16} D^3.$$
$$\therefore \text{when } D = 5, \quad W = \tfrac{5}{16} \text{ of } 125 = 39\tfrac{1}{16}.$$

EXERCISE 125.

1. If $x \propto y$, and if $y = 3$ when $x = 5$, find x when y is 5.

2. If W varies inversely as P, and W is 4 when P is 15, find W when P is 12.

3. If $x \propto y$ and $y \propto z$, show that $xz \propto y^2$.

4. If $x \propto \frac{1}{y}$ and $y \propto \frac{1}{z}$, show that $x \propto z$.

5. If x varies inversely as $y^2 - 1$, and is equal to 24 when $y = 10$, find x when $y = 5$.

6. If x varies as $\frac{1}{y} + \frac{1}{z}$, and is equal to 3 when $y = 1$ and $z = 2$, show that $xyz = 2(y + z)$.

7. If $x - y$ varies inversely as $z + \frac{1}{z}$, and $x + y$ varies inversely as $z - \frac{1}{z}$, find the relation between x and z if $x = 1$, $y = 3$, when $z = \frac{1}{2}$.

8. The area of a circle varies as the square of its radius, and the area of a circle whose radius is 1 foot is 3.1416 square feet. Find the area of a circle whose radius is 20 feet.

9. The volume of a sphere varies as the cube of its radius, and the volume of a sphere whose radius is 1 foot is 4.1888 cubic feet. Find the volume of a sphere whose radius is 2 feet.

10. If a sphere of given material 3 inches in diameter weighs 24 pounds, how much will a sphere of the same material weigh if its diameter is 5 inches?

11. The *velocity* of a falling body varies as the time during which it has fallen from rest. If the velocity of a falling body at the end of 2 seconds is 64 feet, what is its velocity at the end of 8 seconds?

12. The *distance* a body falls from rest varies as the square of the time it is falling. If a body falls through 144 feet in 3 seconds, how far will it fall in 5 seconds?

The volume of a right circular cone varies jointly as its height and the square of the radius of its base.

13. Compare the volume of two cones, one of which is twice as high as the other, but with one half its diameter.

If the volume of a cone 7 feet high with a base whose radius is 3 feet is 66 cubic feet:

14. Find the volume of a cone 9 feet high with a base whose radius is 3 feet.

15. Find the volume of a cone 7 feet high with a base whose radius is 4 feet.

16. Find the volume of a cone 9 feet high with a base whose radius is 4 feet.

17. The volume of a sphere varies as the cube of its radius. If the volume is $179\frac{2}{3}$ cubic feet when the radius is $3\frac{1}{2}$ feet, find the volume when the radius is 1 foot 6 inches.

18. Find the radius of a sphere whose volume is the sum of the volumes of two spheres with radii $3\frac{1}{2}$ feet and 6 feet, respectively.

19. The distance of the offing at sea varies as the square root of the height of the eye above the sea level, and the distance is 3 miles when the height is 6 feet. Find the distance when the height is 24 feet.

CHAPTER XXII.

PROGRESSIONS.

355. A succession of numbers that proceed according to some fixed law is called a **series**; the successive numbers are called the **terms** of the series.

A series that ends at some particular term is a **finite series**; a series that continues without end is an **infinite series.**

Arithmetical Progression.

356. A series is called an **arithmetical series** or an **arithmetical progression** when each term after the first is obtained by adding to the preceding term a *constant difference.*

The general representative of such a series is

$$\begin{array}{cccc} \text{1st} & \text{2d} & \text{3d} & \text{4th} \\ a, & a+d, & a+2d, & a+3d\cdots\cdots, \end{array}$$

in which a is the first term and d the common difference; the series is *increasing* or *decreasing* according as d is *positive* or *negative.*

357. The nth Term. Since each succeeding term of the series is obtained by adding d to the preceding term, the coefficient of d is always one less than the number of the term, so that the nth term is $a+(n-1)d$.

If the nth term is represented by l, we have

$$l = a + (n-1)d.$$

358. Sum of the Series. If l denotes the nth term, a the first term, n the number of terms, d the common difference, and s the sum of n terms, it is evident that

$$\begin{array}{rl} s = & a + (a+d) + (a+2d) + \cdots + (l-d) + l, \quad \text{or} \\ s = & \underline{l + (l-d) + (l-2d) + \cdots + (a+d) + a,} \\ \therefore 2s = & (a+l) + (a+l) + (a+l) + \cdots + (a+l) + (a+l) \\ = & n(a+l). \\ \therefore s = & \frac{n}{2}(a+l). \end{array}$$

359. From the equations

I, $$\mathbf{l = a + (n-1)\,d},$$

II, $$\mathbf{s = \frac{n}{2}(a + l)},$$

any *two* of the five numbers a, d, l, n, s may be found when the other *three* are given.

1. Find the thirteenth term of an arithmetical progression, if the first term is 3 and the common difference 5.

Here $a = 3,\ d = 5,\ n = 13.$

From I, $l = 3 + (13 - 1)5 = 63.$

2. Find the arithmetical series, if the tenth term is 31 and the twentieth term 61.

$$\begin{array}{lr} \text{From I,} & a + 19d = 61 \\ \text{and} & \underline{a + \ \ 9d = 31} \\ \text{Subtract} & 10d = 30 \\ \text{Whence} & d = 3. \\ \text{Therefore,} & a = 4. \end{array}$$

Therefore, the series is 4, 7, 10 ....

3. Find the sum of ten terms of the series 2, 5, 8, 11 .....

Here $a = 2,\ d = 3,\ n = 10.$

From I, $l = 2 + 27 = 29.$

Substitute in II, $s = \frac{10}{2}(2 + 29) = 155.$

The sum of ten terms is 155.

4. The first term of an arithmetical series is 3, the last term 31, and the sum of the series 136. Find the series.

From I, $31 = 3 + (n - 1)\,d.$ (1)

From II, $136 = \frac{n}{2}(3 + 31).$ (2)

From (2), $n = 8.$

Substitute in (1), $d = 4.$

The series is 3, 7, 11, 15, 19, 23, 27, 31.

5. How many terms of the series 5, 9, 13 ..... must be taken in order that their sum may be 275?

From I, $l = 5 + (n - 1)4.$

$\therefore l = 4n + 1.$ (1)

From II, $275 = \frac{n}{2}(5 + l).$ (2)

Substitute in (2) the value of l found in (1),

$$275 = \frac{n}{2}(4n + 6),$$

or $2n^2 + 3n = 275.$

Complete the square,

$$16n^2 + (\quad) + 9 = 2209.$$

Extract the root, $4n + 3 = \pm 47.$

Therefore, $n = 11, \text{ or } -12\tfrac{1}{2}.$

We use only the positive result.

Therefore, 11 terms must be taken.

6. Find n when d, l, s are given.

From I, $$a = l - (n - 1)d.$$

From II, $$a = \frac{2s - ln}{n}.$$

Therefore, $$l - (n - 1)d = \frac{2s - ln}{n}.$$

$$ln - dn^2 + dn = 2s - ln.$$

$$dn^2 - (2l + d)n = -2s.$$

Complete the square,

$$4d^2n^2 - (\;\;) + (2l + d)^2 = (2l + d)^2 - 8ds.$$

Extract the root,

$$2dn - (2l + d) = \pm\sqrt{(2l + d)^2 - 8ds}.$$

$$\therefore n = \frac{2l + d \pm \sqrt{(2l + d)^2 - 8ds}}{2d}.$$

360. The **arithmetical mean** between two numbers is the number which stands between them, and makes with them an arithmetical series.

If a and b represent two numbers, and A their arithmetical mean, then a, A, b are in arithmetical progression.

$$\therefore A - a = d, \qquad \text{and } b - A = d.$$

$$\therefore A - a = b - A.$$

$$\therefore A = \frac{a + b}{2}.$$

361. Sometimes it is required to insert several arithmetical means between two numbers.

Insert six arithmetical means between 3 and 17.

Here the whole number of terms is eight; therefore, by I,

$$17 = 3 + 7d.$$

$$\therefore d = 2.$$

The series is 3, [5, 7, 9, 11, 13, 15,] 17, the terms in brackets being the means required.

NOTE. The student should work out all the problems on the following page, using the formulas I and II.

No.	Given.	Required.	Formulas.
1	$a\ d\ n$	l	$l = a + (n - 1)\,d.$
2	$a\ d\ s$		$l = -\tfrac{1}{2}d \pm \sqrt{2\,ds + (a - \tfrac{1}{2}d)^2}.$
3	$a\ n\ s$		$l = \dfrac{2\,s}{n} - a.$
4	$d\ n\ s$		$l = \dfrac{s}{n} + \dfrac{(n-1)\,d}{2}.$
5	$a\ d\ n$	s	$s = \tfrac{1}{2}n\,[2\,a + (n-1)\,d].$
6	$a\ d\ l$		$s = \dfrac{l + a}{2} + \dfrac{l^2 - a^2}{2\,d}.$
7	$a\ n\ l$		$s = (l + a)\,\dfrac{n}{2}.$
8	$d\ n\ l$		$s = \tfrac{1}{2}n\,[2\,l - (n-1)\,d].$
9	$d\ n\ l$	a	$a = l - (n-1)\,d.$
10	$d\ n\ s$		$a = \dfrac{s}{n} - \dfrac{(n-1)\,d}{2}.$
11	$d\ l\ s$		$a = \tfrac{1}{2}d \pm \sqrt{(l + \tfrac{1}{2}d)^2 - 2\,ds}.$
12	$n\ l\ s$		$a = \dfrac{2\,s}{n} - l.$
13	$a\ n\ l$	d	$d = \dfrac{l - a}{n - 1}.$
14	$a\ n\ s$		$d = \dfrac{2\,(s - an)}{n\,(n-1)}.$
15	$a\ l\ s$		$d = \dfrac{l^2 - a^2}{2\,s - l - a}.$
16	$n\ l\ s$		$d = \dfrac{2\,(ln - s)}{n\,(n-1)}.$
17	$a\ d\ l$	n	$n = \dfrac{l - a}{d} + 1.$
18	$a\ d\ s$		$n = \dfrac{d - 2\,a \pm \sqrt{(2\,a - d)^2 + 8\,ds}}{2\,d}.$
19	$a\ l\ s$		$n = \dfrac{2\,s}{l + a}.$
20	$d\ l\ s$		$n = \dfrac{2\,l + d \pm \sqrt{(2\,l + d)^2 - 8\,ds}}{2\,d}.$

EXERCISE 126.

1. Find the tenth term of 9, 13, 17 .....

2. Find the thirteenth term of $-3, -1, +1$ .....

3. Find the ninth term of $-5, -8, -11$ .....

4. Find the eighth term of a, $a+3b$, $a+6b$ .....

5. Find the fifteenth term of $1, \frac{6}{7}, \frac{5}{7}$ .....

6. Find the fourteenth term of $-44, -40, -36$ .....

7. The first term of an arithmetical series is 3, the thirteenth term is 55. Find the common difference.

8. Find the arithmetical mean between -5 and 17; between a^2+ab+b^2 and a^2-ab+b^2.

9. Insert three arithmetical means between 1 and 19; and four between -4 and 17.

10. The first term of a series is 2, and the common difference $\frac{1}{3}$. What term will be 12?

11. The seventh term of a series, whose common difference is 3, is 11. Find the first term.

Find the sum of:

12. $5+8+11+$ ..... to ten terms.

13. $-4-1+2+$ ..... to seven terms.

14. $a+4a+7a+$ ..... to n terms.

15. $\frac{2}{3}+\frac{7}{15}+\frac{4}{15}+$ ..... to twenty-one terms.

16. $1+2\frac{2}{3}+4\frac{1}{3}+$ ..... to twenty terms.

17. The sum of six terms of an arithmetical series is 27, and the first term is 1. Find the series.

18. How many terms of the series $-5, -2, +1$ ..... must be taken that their sum may be 63?

19. The first term of an arithmetical series is 12, and the sum of ten terms is 10. Find the tenth term.

20. When a train arrives at the top of a long slope, the last car is detached and begins to descend, passing over 3 feet in the first second, three times 3 feet in the second second, five times 3 feet in the third second, etc. At the end of 30 seconds it reaches the bottom of the slope. Find its velocity the last second.

21. Insert eleven arithmetical means between 1 and 12.

22. The first term of an arithmetical series is 3, and the sum of six terms is 28. What term will be 9?

23. The arithmetical mean between two numbers is 10, and the mean between the double of the first and the triple of the second is 27. Find the numbers.

24. The first term of an arithmetical progression is 3, and the third term is 11. Find the sum of seven terms.

25. A common clock strikes the hours from 1 to 12. How many times does it strike every 24 hours?

26. The Greenwich clock strikes the hours from 1 to 24. How many times does it strike in 24 hours?

27. Find three numbers in arithmetical progression of which the sum is 21, and the sum of the first and second $\frac{3}{4}$ of the sum of the second and third.

Let $x - y$, x, and $x + y$ stand for the numbers.

28. The sum of three numbers in arithmetical progression is 33, and the sum of their squares is 461. Find the numbers.

29. The sum of four numbers in arithmetical progression is 12, and the sum of their squares is 116. What are the numbers?

Let $x - 3y$, $x - y$, $x + y$, and $x + 3y$ stand for the numbers.

Geometrical Progression.

362. A series is called a **geometrical series** or a **geometrical progression** when each succeeding term is obtained by multiplying the preceding term by a *constant multiplier.*

The general representative of such a series is

$$a,\ ar,\ ar^2,\ ar^3,\ ar^4 \cdots\cdots,$$

in which a is the first term and r the constant multiplier, or ratio.

The terms increase or decrease in numerical magnitude according as r is numerically greater than or numerically less than unity.

363. The nth Term. Since the exponent of r increases by one for each succeeding term after the first, the exponent is always one less than the number of the term, so that the *nth* term is ar^{n-1}.

If the *nth* term is represented by l, we have

I, $$\mathbf{l = ar^{n-1}}.$$

364. Sum of the Series. If l represents the **nth** term, a the first term, n the number of terms, r the common ratio, and s the sum of n terms, then,

$$s = a + ar + ar^2 + \cdots\cdots + ar^{n-1}. \qquad (1)$$

Multiply by r,

$$rs = ar + ar^2 + ar^3 + \cdots\cdots + ar^{n-1} + ar^n. \qquad (2)$$

Subtract (1) from (2),

$$rs - s = ar^n - a,$$

or $$(r-1)s = a(r^n - 1).$$

II, $$\therefore \mathbf{s = \frac{a(r^n - 1)}{r - 1}}.$$

365. When $r < 1$, formula II may be more conveniently written

$$s = \frac{a(1 - r^n)}{1 - r}.$$

366. $l = ar^{n-1}$. (§ 363)

Multiply by r, $rl = ar^n$.

By putting rl for ar^n in formula II, we have

III, $$\mathbf{s = \frac{rl - a}{r - 1}}.$$

When $r < 1$, formula III is more conveniently written

$$s = \frac{a - rl}{1 - r}.$$

367. From the two formulas I and II, or the two formulas I and III, any *two* of the five numbers a, r, l, n, s may be found when the other *three* are given.

1. Find the tenth term of a geometrical series if the first term is 3 and the ratio 2.

Here $a = 3,\ r = 2,\ n = 10.$

From I, $l = 3 \times 2^9 = 3 \times 512 = 1536.$

2. Find the geometrical series if the third term is 20 and the sixth term 160.

Let $a =$ the first term, and $r =$ the ratio.

Then, $ar^2 =$ the third term, and $ar^5 =$ the sixth term.

Therefore, $\frac{ar^5}{ar^2} = \frac{160}{20}.$

$r^3 = 8.$

$\therefore r = 2.$

Since $ar^2 = 20,\ a = 20 \div 4 = 5.$

The series is 5, 10, 20, 40 .....

3. Find the sum of six terms of the series 3, 6, 12 .....

Here $a = 3,\ r = 2,\ n = 6.$

From II, $$s = \frac{a(r^n - 1)}{r - 1} = \frac{3(2^6 - 1)}{r - 1} = 3 \times 63 = 189.$$

4. The first term of a geometrical series is 3, the last term 192, and the sum of the series 381. Find the number of terms and the ratio.

From I, $192 = 3r^{n-1}$. (1)

From III, $381 = \dfrac{192r - 3}{r - 1}$. (2)

From (2), $r = 2$.

Substitute in (1), $2^{n-1} = 64$.

$\therefore n = 7$.

The series is 3, 6, 12, 24, 48, 96, 192.

368. The **geometrical mean** between two numbers is the number which stands between them, and makes with them a geometrical series.

If a and b denote two numbers, and G their geometrical mean, then a, G, b are in geometrical progression, and by the definition of a geometrical series (§ 362),

$$\frac{G}{a} = r, \text{ and } \frac{b}{G} = r.$$

$$\therefore \frac{G}{a} = \frac{b}{G}.$$

$$\therefore G = \sqrt{ab}.$$

369. Sometimes it is required to insert several geometrical means between two numbers.

Insert three geometrical means between 3 and 48.

Here the whole number of terms is five; 3 is the first term and 48 the fifth.

By I, $48 = 3r^4$.

$r^4 = 16$.

$\therefore r = \pm 2$.

The series is $3,\ [\ 6,\ 12,\ 24],\ 48$;

or $3,\ [-6,\ 12,\ -24],\ 48$.

The terms in brackets are the means required

In working out the following results, the student should make use of formulas I, II, and III.

No.	Given.	Required.	Formulas.
1	$a\ r\ n$	l	$l = ar^{n-1}.$
2	$a\ r\ s$		$l = \frac{a + (r-1)s}{r}.$
3	$a\ n\ s$		$l(s-l)^{n-1} - a(s-a)^{n-1} = 0.$
4	$r\ n\ s$		$l = \frac{(r-1)sr^{n-1}}{r^n - 1}.$
5	$a\ r\ n$	s	$s = \frac{a(r^n - 1)}{r - 1.}.$
6	$a\ r\ l$		$s = \frac{rl - a}{r - 1}.$
7	$a\ n\ i$		$s = \frac{\sqrt[n-1]{l^n} - \sqrt[n-1]{a^n}}{\sqrt[n-1]{l} - \sqrt[n-1]{a}}.$
8	$r\ n\ l$		$s = \frac{lr^n - l}{r^n - r^{n-1}}.$
9	$r\ n\ l$	a	$a = \frac{l}{r^{n-1}}.$
10	$r\ n\ s$		$a = \frac{(r-1)s}{r^n - 1}.$
11	$r\ l\ s$		$a = rl - (r-1)s.$
12	$n\ l\ s$		$a(s-a)^{n-1} - l(s-l)^{n-1} = 0.$
13	$a\ n\ l$	r	$r = \sqrt[n-1]{\frac{l}{a}}.$
14	$a\ n\ s$		$r^n - \frac{s}{a}r + \frac{s-a}{a} = 0.$
15	$a\ l\ s$		$r = \frac{s-a}{s-l}.$
16	$n\ l\ s$		$r^n - \frac{s}{s-l}r^{n-1} + \frac{l}{s-l} = 0.$

EXERCISE 127.

1. Find the seventh term of 2, 6, 18 .....

2. Find the sixth term of 3, 6, 12 .....

3. Find the ninth term of 6, 3, $1\frac{1}{2}$ .....

4. Find the eighth term of 1, -2, 4 .....

5. Find the fifth term of $4a$, $-6ma^2$, $9m^2a^3$ .....

6. Find the geometrical mean between $18x^3y$ and $30xy^3z$.

7. Find the ratio when the first and third terms are 5 and 80, respectively.

8. Insert two geometrical means between 8 and 125; and three between 14 and 224.

9. If $a = 2$ and $r = 3$, which term will be equal to 162?

10. The fifth term of a geometrical series is 48, and the ratio 2. Find the first and seventh terms.

Find the sum of:

11. $3 + 6 + 12 + \cdots\cdots$ to eight terms.

12. $1 - 3 + 9 - \cdots\cdots$ to seven terms.

13. $8 + 4 + 2 + \cdots\cdots$ to ten terms.

14. $0.1 + 0.5 + 2.5 + \cdots\cdots$ to seven terms.

15. $m - \frac{m}{4} + \frac{m}{16} - \cdots\cdots$ to five terms.

16. The population of a city increased in four years from 10,000 to 14,641. What was the annual rate of increase?

17. The sum of four numbers in geometrical progression is 200, and the first term is 5. Find the ratio.

18. Find the sum of eight terms of a geometrical series whose last term is 1, and fifth term $\frac{1}{8}$.

19. In an odd number of terms, show that the product of the first and last is equal to the square of the middle term.

20. The product of four terms of a geometrical series is 4, and the fourth term is 4. Find the series.

21. If from a line one third is cut off, then one third of the remainder, and so on, what fraction of the whole will remain when this has been done five times?

22. Of three numbers in geometrical progression, the sum of the first and second exceeds the third by 3, and the sum of the first and third exceeds the second by 21. What are the numbers?

23. Find two numbers whose sum is $3\frac{1}{4}$ and geometrical mean $1\frac{1}{2}$.

24. The sum of the squares of two numbers exceeds twice their product by 576; the arithmetical mean of the two numbers exceeds the geometrical mean by 6. Find the numbers.

25. There are four numbers such that the sum of the first and the last is 11, and the sum of the other two is 10. The first three of these four numbers are in arithmetical progression, and the last three are in geometrical progression. Find the numbers.

26. Find three numbers in geometrical progression such that their sum is 13 and the sum of their squares 91.

27. The difference between two numbers is 48, and the arithmetical mean exceeds the geometrical mean by 18. Find the numbers.

28. There are four numbers in geometrical progression, the second of which is less than the fourth by 24, and the sum of the extremes is to the sum of the means as 7 to 3. Find the numbers.

Infinite Geometrical Series.

370. When r is less than 1, the successive terms of a geometrical series become numerically smaller; by taking n large enough we can make the nth term, ar^{n-1}, as small as we please, although we cannot make it absolutely zero.

The sum of n terms, $\frac{a-rl}{1-r}$ (§ 366), may be written $\frac{a}{1-r}-\frac{rl}{1-r}$; this sum differs from $\frac{a}{1-r}$ by $\frac{rl}{1-r}$; by taking enough terms we can make l, and consequently $\frac{rl}{1-r}$, as small as we please. Hence, $\frac{a}{1-r}$ may be considered the *sum* of an infinite number of terms of the series.

1. Find the sum of $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\cdots\cdots$

Here $a=1$, and $r=-\frac{1}{2}$.

$$\therefore s=\frac{a}{1-r}=\frac{1}{1-(-\frac{1}{2})}=\frac{1}{1+\frac{1}{2}}=\frac{2}{3}.$$

2. Find the value of $0.2363636\cdots\cdots$

The terms after the first form a decreasing geometrical series in which $a=0.036$, and $r=0.01$.

$$\therefore s=\frac{a}{1-r}=\frac{0.036}{1-0.01}=\frac{0.036}{0.99}=\frac{36}{990}=\frac{2}{55}.$$

Therefore, the required value is $\frac{2}{10}+\frac{2}{55}=\frac{22+4}{110}=\frac{13}{55}.$

Exercise 128.

Find the sum of the following infinite series:

1. $4+2+1+\cdots\cdots$
2. $\frac{1}{2}+\frac{1}{3}+\frac{2}{9}+\cdots\cdots$
3. $\frac{1}{4}-\frac{1}{16}+\frac{1}{64}-\cdots\cdots$
4. $1-\frac{2}{5}+\frac{4}{25}-\cdots\cdots$
5. $2-1\frac{1}{3}+\frac{8}{9}-\cdots\cdots$
6. $0.1+0.01+0.001+\cdots\cdots$
7. $0.868686\cdots\cdots$
8. $0.54444\cdots\cdots$

* Harmonical Progression.

371. A series is called a **harmonical series** or a **harmonical progression** when the *reciprocals* of its terms form an *arithmetical series.*

Hence, the general representative of such a series is

$$\frac{1}{a},\ \frac{1}{a+d},\ \frac{1}{a+2d},\ \ldots\ldots,\ \frac{1}{a+(n-1)d}.$$

372. Questions relating to harmonical series should be solved by writing the reciprocals of its terms so as to form an arithmetical series.

373. If a and b denote two numbers, and H their harmonical mean, then, by the definition of a harmonical series,

$$\frac{1}{H}-\frac{1}{a}=\frac{1}{b}-\frac{1}{H}.$$

$$\therefore \frac{2}{H}=\frac{1}{a}+\frac{1}{b}=\frac{a+b}{ab}.$$

$$\therefore H=\frac{2ab}{a+b}.$$

374. Sometimes it is required to insert several harmonical means between two numbers.

Let it be required to insert three harmonical means between 3 and 18.

Find the three arithmetical means between $\frac{1}{3}$ and $\frac{1}{18}$.

These are found to be $\frac{19}{72}$, $\frac{14}{72}$, $\frac{9}{72}$; therefore, the harmonical means are $\frac{72}{19}$, $\frac{72}{14}$, $\frac{72}{9}$; or $3\frac{15}{19}$, $5\frac{1}{7}$, 8.

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* A harmonical series is so called because musical strings of uniform thickness and tension produce *harmony* when their lengths are represented by the *reciprocals* of the natural series of numbers; that is, by the harmonical series 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, etc.

EXERCISE 129.

1. Insert four harmonical means between 2 and 12.

2. Find two numbers whose difference is 8 and harmonical mean $1\frac{4}{5}$.

3. Find the seventh term of the harmonical series 3, $3\frac{3}{7}$, 4 .....

4. Continue to two terms each way the harmonical series two consecutive terms of which are 15, 16.

5. The first two terms of a harmonical series are 5 and 6. Which term will equal 30?

6. The fifth and ninth terms of a harmonical series are 8 and 12. Find the first four terms.

7. The difference between the arithmetical and harmonical means between two numbers is $1\frac{4}{5}$, and one of the numbers is four times the other. Find the numbers.

8. Find the arithmetical, geometrical, and harmonical means between two numbers, a and b; and show that the geometrical mean is a mean proportional between the arithmetical and harmonical means. Also, arrange these means in order of magnitude.

9. The arithmetical mean between two numbers exceeds the geometrical by 13, and the geometrical exceeds the harmonical by 12. What are the numbers?

10. The sum of three terms of a harmonical series is 11, and the sum of their squares is 49. Find the numbers.

11. When a, b, c are in harmonical progression, show that $a : c :: a - b : b - c$.

CHAPTER XXIII.

VARIABLES AND LIMITS.

375. Constants and Variables. A number that, under the conditions of the problem into which it enters, may be made to assume any one of an unlimited number of values is called a *variable.*

A number that, under the conditions of the problem into which it enters, has a fixed value is called a *constant.*

Variables are represented by x, y, z; constants by a, b, c, and by the Arabic numerals.

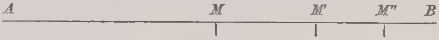
376. Limits. When the value of a variable, measured at a series of definite intervals, can by continuing the series be made to differ from a given constant by less than any assigned quantity, however small, but cannot be made absolutely equal to the constant, the constant is called the *limit* of the variable, and the variable is said to *approach indefinitely to its limit.*

Consider the repetend 0.333....., which may be written $\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \cdots\cdots$

The value of each fraction after the first is one tenth of the preceding fraction, and by continuing the series we shall reach a fraction less than *any* assigned value, *however small*; that is, the values of the successive fractions *approach 0 as a limit.*

The *sum* of these fractions will always be less than $\frac{1}{3}$; but the more terms we take, the nearer does the sum *approach $\frac{1}{3}$ as a limit.*

Suppose a point to move from A toward B, under the conditions that the first second it shall move one half the distance from A to B, that is, to M; the next second, one half the remaining distance, that is, to M'; the next second, one half the remaining distance, that is, to M''; and so on indefinitely.



Then it is evident that the moving point *may approach as near to B as we please, but will never arrive at B.* For, however near it may be to B at any instant, the next second it will pass over one half the interval still remaining; it must, therefore, approach nearer to B, since *half* the interval still remaining is *some* distance, but will not reach B since *half* the interval still remaining is not the *whole* distance.

Hence, the distance from A to the moving point is an increasing variable, which indefinitely approaches the constant AB as its *limit;* and the distance from the moving point to B is a decreasing variable, which indefinitely approaches the constant zero as its *limit.*

If the length of AB is two inches, and the variable is denoted by x, and the difference between the variable and its limit by y:

after one second, $x = 1$, $y = 1$;
after two seconds, $x = 1 + \frac{1}{2}$, $y = \frac{1}{2}$;
after three seconds, $x = 1 + \frac{1}{2} + \frac{1}{4}$, $y = \frac{1}{4}$;
after four seconds, $x = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$, $y = \frac{1}{8}$;
and so on indefinitely.

Now the sum of the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$, etc., is less than 2; but by taking a great number of terms, the sum can be made to differ from 2 by as little as we please. Hence, 2 is the limit of the sum of the series, when the number of the terms is increased indefinitely; and 0 is the limit of the difference between this variable sum and 2.

377. Test for a Limit. In order to prove that a variable approaches a constant as a limit, it is *necessary* and *sufficient* to prove that the difference between the variable and the constant can be made *as near to 0 as we please,* but cannot be made *absolutely equal to 0.*

A variable may approach a constant without approaching it *as a limit.* Thus, in the last example x approaches 3, but not as a limit; for $3 - x$ cannot be made as near to 0 as we please, since it cannot be made less than 1.

378. Infinites. As a variable changes its value, it may constantly increase in numerical value; if the variable can become numerically greater than any assigned value, *however great* this assigned value may be, the variable is said to *increase without limit,* or to *increase indefinitely.*

When a variable is conceived to have a value greater than any assigned value, however great, the variable is said to become *infinite;* such a variable is called an *infinite number,* or simply an **infinite,** and is denoted by ∞.

379. Infinitesimals. As a variable changes its value, it may constantly decrease in numerical value; if the variable can become numerically less than any assigned value, *however small* this assigned value may be, the variable is said to *decrease without limit,* or to *decrease indefinitely.*

In this case the variable approaches 0 as a limit.

When a variable which approaches 0 as a limit is conceived to have a value less than any assigned value, however small this assigned value may be, the variable is said to become *infinitesimal;* such a variable is called an *infinitesimal number,* or simply an **infinitesimal.**

380. Infinites and infinitesimals are *variables,* not constants. There is no idea of *fixed* value implied in either an infinite or an infinitesimal.

381. An infinitesimal *is not 0.* An infinitesimal is a variable arising from *the division of a quantity into a constantly increasing number of parts;* 0 is a constant arising from taking *the difference of two equal quantities.*

382. Finite Numbers. A number which cannot become infinite is said to be **finite.**

Theorems of Infinites and Infinitesimals.

383. THEOREM 1. *If $\boldsymbol{x}$ is infinitesimal, and $\boldsymbol{a}$ is finite and not 0, then $\boldsymbol{ax}$ is infinitesimal.*

For ax can be made as small as we please since x can be made as small as we please.

384. THEOREM 2. *If $\boldsymbol{X}$ is infinite, and $\boldsymbol{a}$ is finite and not 0, then $\boldsymbol{aX}$ is infinite.*

For aX can be made as large as we please since X can be made as large as we please.

385. THEOREM 3. *If $\boldsymbol{x}$ is infinitesimal, and $\boldsymbol{a}$ is finite and not 0, then $\boldsymbol{\frac{a}{x}}$ is infinite.*

For $\frac{a}{x}$ can be made as large as we please since x can be made as small as we please.

386. THEOREM 4. *If $\boldsymbol{X}$ is infinite, and $\boldsymbol{a}$ is finite and not 0, then $\boldsymbol{\frac{a}{X}}$ is infinitesimal.*

For $\frac{a}{X}$ can be made as small as we please since X can be made as large as we please.

In the above theorems a may be a constant or a variable; the only restriction on the value of a is that it shall not become either infinite or 0.

387. Abbreviated Notation. The expression $\frac{a}{0}$ cannot be interpreted literally, since we cannot divide by 0; neither can $\frac{a}{\infty} = 0$ be interpreted literally, since we can find no number such that the quotient obtained by dividing a by that number is 0.

The expression $\frac{a}{0} = \infty$ is simply an abbreviated way of writing:

If $\frac{a}{x} = X$, *and* x *approaches 0 as a limit,* X *increases without limit.*

The expression $\frac{a}{\infty} = 0$ is simply an abbreviated way of writing:

If $\frac{a}{X} = x$, *and* X *increases without limit,* x *approaches 0 as a limit.*

388. Approach to a Limit. A variable may approach its limit in one of three ways:

1. The variable may be always less than its limit.
2. The variable may be always greater than its limit.
3. The variable may be sometimes less and sometimes greater than its limit.

If x represent the sum of n terms of the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$, x is always less than its limit 2.

If x represent the sum of n terms of the series $3 - \frac{1}{2} - \frac{1}{4} - \frac{1}{8} - \cdots$, x is always greater than its limit 2.

If x represent the sum of n terms of the series $3 - \frac{3}{2} + \frac{3}{4} - \frac{3}{8} + \cdots$, we have (§ 364)

$$x = \frac{3 - 3\left(-\frac{1}{2}\right)^n}{1 + \frac{1}{2}} = 2 - 2\left(-\frac{1}{2}\right)^n.$$

As n is indefinitely increased, x evidently approaches 2 as a limit.

If n is even, x is less than 2; if n is odd, x is greater than 2.

Hence, if n is increased by taking each time one more term, x will be alternately less than and greater than 2. If, for example,

$$n = 2, \quad 3, \quad 4, \quad 5, \quad 6, \quad 7,$$

$$x = 1\tfrac{1}{2}, \quad 2\tfrac{1}{4}, \quad 1\tfrac{7}{8}, \quad 2\tfrac{1}{16}, \quad 1\tfrac{31}{32}, \quad 2\tfrac{63}{64}.$$

Indeterminate Forms. — Vanishing Fractions.

389. When one or more variables are involved in both numerator and denominator of a fraction, it may happen that for certain values of the variables both numerator and denominator of the fraction vanish. The fraction then assumes the indeterminate form $\frac{0}{0}$. If there is but *one* variable involved, we may obtain a definite value as follows:

Let x be the variable, and a the value of x for which the fraction assumes the form $\frac{0}{0}$. Give to x a value a little greater than a, as $a + h$; the fraction will now have a definite value. The limit of this last value, as h is indefinitely decreased, is called the **limiting value** of the fraction.

The fundamental indeterminate form is $\frac{0}{0}$, and all other indeterminate forms may be reduced to this.

Thus,

$$\frac{\infty}{\infty} = \frac{a}{0} \div \frac{b}{0} = \frac{a}{0} \times \frac{0}{b} = \frac{0}{0}.$$

$$0 \times \infty = 0 \times \frac{a}{0} = \frac{0 \times a}{0} = \frac{0}{0}.$$

$$\infty - \infty = \frac{a}{0} - \frac{b}{0} = \frac{0 \times a - 0 \times b}{0} = \frac{0}{0}.$$

1. Find the limiting value of $\frac{x^2 - a^2}{x - a}$ as x approaches a.

When x has the value a, the fraction assumes the form $\frac{0}{0}$.

Put $x = a + h$; the fraction becomes

$$\frac{(a+h)^2 - a^2}{(a+h) - a} = \frac{2ah + h^2}{h}.$$

Since h is not 0, we can divide by h and obtain $2a + h$.

As h is indefinitely decreased, this approaches $2a$ as a limit.

2. Find the limiting value of $\frac{2x^3 - 4x + 5}{3x^3 + 2x^2 - 1}$ when x becomes infinite.

Divide each term of the numerator and denominator by x^3. Then,

$$\frac{2x^3 - 4x + 5}{3x^3 + 2x^2 - 1} = \frac{2 - \frac{4}{x^2} + \frac{5}{x^3}}{3 + \frac{2}{x} - \frac{1}{x^3}}.$$

As x increases indefinitely, each term that contains x of the last fraction approaches 0 as a limit (Theorem 4), and the fraction approaches $\frac{2}{3}$ as a limit.

Exercise 130.

Find the limiting value of:

1. $\frac{(4x^2 - 3)(1 - 2x)}{7x^3 - 6x + 4}$ when x becomes infinitesimal.
2. $\frac{(x^2 - 5)(x^2 + 7)}{x^4 + 35}$ when x becomes infinite.
3. $\frac{(x + 2)^3}{x^2 + 4}$ when x becomes infinitesimal.
4. $\frac{x^2 - 8x + 15}{x^2 - 7x + 12}$ when x approaches 3.
5. $\frac{x^2 - 9}{x^2 + 9x + 18}$ when x approaches -3.
6. $\frac{x(x^2 + 4x + 3)}{x^3 + 3x^2 + 5x + 3}$ when x approaches -1.
7. $\frac{x^3 + x^2 - 2}{x^3 + 2x^2 - 2x - 1}$ when x approaches 1.
8. $\frac{4x + \sqrt{x - 1}}{2x - \sqrt{x + 1}}$ when x approaches 1.

CHAPTER XXIV.

PROPERTIES OF SERIES.

Convergent and Divergent Series.

390. By performing the indicated division, we obtain from $\frac{1}{1-x}$ the infinite series $1 + x + x^2 + x^3 + \cdots\cdots$ This series, however, is not equal to the fraction for all values of x.

391. If x is numerically less than 1, the series is equal to the fraction. In this case we can obtain an approximate value for the sum of the series by taking the sum of a number of terms; the greater the number of terms taken, the nearer will this approximate sum approach the value of the fraction. The approximate sum will never be exactly equal to the fraction, however great the number of terms taken; but by taking enough terms it can be made to differ from the fraction by as little as we please.

Thus, if $x = \frac{1}{2}$, the value of the fraction is 2, and the series is

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots\cdots$$

The sum of four terms of this series is $1\frac{7}{8}$; the sum of five terms, $1\frac{15}{16}$; the sum of six terms, $1\frac{31}{32}$; and so on. The successive approximate sums approach, but never reach, the finite value 2.

392. An infinite series is said to be **convergent** when the sum of the terms, as the number of terms is indefinitely increased, *approaches some fixed finite value;* this finite value is called the **sum of the series.**

393. In the series $1 + x + x^2 + x^3 + \cdots$ suppose x numerically greater than 1. In this case the greater the number of terms taken, the greater will their sum be; by taking enough terms we can make their sum as large as we please. The fraction, on the other hand, has a definite value. Hence, when x is numerically greater than 1, the series is *not* equal to the fraction.

Thus, if $x = 2$, the value of the fraction is -1, and the series is

$$1 + 2 + 4 + 8 + \cdots$$

The greater the number of terms taken, the larger the sum. Evidently, the fraction and the series are not equal.

394. In the same series suppose $x = 1$. In this case the fraction is $\frac{1}{1-1} = \frac{1}{0}$, and the series $1 + 1 + 1 + 1 + \cdots$ The more terms we take, the greater will the sum of the series be, and the sum of the series does *not* approach a *fixed finite value.*

If x, however, is not exactly 1, but is a little less than 1, the value of the fraction $\frac{1}{1-x}$ will be very great, and the fraction will be equal to the series.

Suppose $x = -1$. In this case the fraction is $\frac{1}{1+1} = \frac{1}{2}$, and the series $1 - 1 + 1 - 1 + \cdots$ If we take an even number of terms, their sum is 0; if an odd number, their sum is 1. Hence, the fraction is *not* equal to the series.

395. A series is said to be **divergent** when the sum of the terms, as the number of terms is indefinitely increased, either increases without end, or oscillates in value *without approaching any fixed finite value.*

No reasoning can be based on a divergent series; hence, in using an infinite series it is necessary to make such restrictions as will cause the series to be convergent. Thus, we can use the infinite series $1 + x + x^2 + x^3 + \cdots\cdots$ when, and only when, x lies between $+1$ and -1.

396. THEOREM. *If two series, arranged by powers of x, are equal for all values of x that make both series convergent, the corresponding coefficients are equal each to each.*

For, if $A + Bx + Cx^2 + \cdots\cdots = A' + B'x + C'x^2 + \cdots\cdots$,

by transposition,

$$A - A' = (B' - B)x + (C' - C)x^2 + \cdots\cdots$$

Now, by taking x sufficiently small, the right side of this equation can be made *less* than any assigned value whatever, and therefore less than $A - A'$, if $A - A'$ has any value whatever. Hence, $A - A'$ cannot have any value.

Therefore, $\quad A - A' = 0$, or $A = A'$.

Hence, $Bx + Cx^2 + Dx^3 + \cdots\cdots = B'x + C'x^2 + D'x^3 + \cdots\cdots$,

or $\quad (B - B')x = (C' - C)x^2 + (D' - D)x^3 + \cdots\cdots$

Divide by x,

$$B - B' = (C' - C)x + (D' - D)x^2 + \cdots\cdots$$

By the same proof as for $A - A'$,

$$B - B' = 0, \text{ or } B = B'.$$

In like manner,

$$C = C', D = D'; \text{ and so on.}$$

Hence, the equation

$$A + Bx + Cx^2 + \cdots\cdots = A' + B'x + C'x^2 + \cdots\cdots,$$

if true for all finite values of x, is an **identical equation**; that is, *the coefficients of like powers of x are equal.*

Indeterminate Coefficients.

397. Expand $\frac{2+3x}{1+x+x^2}$ in ascending powers of x

Assume $$\frac{2+3x}{1+x+x^2} = A + Bx + Cx^2 + Dx^3 + \cdots\cdots;$$

then, by clearing of fractions,

$$\begin{aligned} 2+3x = A + Bx &+ Cx^2 + Dx^3 + \cdots\cdots \\ + Ax &+ Bx^2 + Cx^3 + \cdots\cdots \\ &+ Ax^2 + Bx^3 + \cdots\cdots \end{aligned}$$

$$\therefore 2+3x = A + (B+A)x + (C+B+A)x^2 + (D+C+B)x^3 + \cdots\cdots$$

By § 396, $A = 2,\ B + A = 3,\ C + B + A = 0,\ D + C + B = 0;$

whence, $B = 1,\ C = -3,\ D = 2;$ and so on.

$$\therefore \frac{2+3x}{1+x+x^2} = 2 + x - 3x^2 + 2x^3 + \cdots\cdots$$

The series is of course equal to the fraction for only such values of x as make the series convergent.

Note. In employing the method of Indeterminate Coefficients, the form of the given expression must determine what powers of the variable x must be assumed. It is necessary and sufficient that the assumed equation, when simplified, shall have in the right member all the powers of x that are found in the left member.

If any powers of x occur in the *right* member that are not in the *left* member, the coefficients of these powers in the right member will vanish, so that in this case the method still applies; but if any powers of x occur in the *left* member that are not in the *right* member, then the coefficients of these powers of x must be put equal to 0 in equating the coefficients of like powers of x; and this leads to absurd results. Thus, if it were assumed that

$$\frac{2+3x}{1+x+x^2} = Ax + Bx^2 + Cx^3 + \cdots\cdots,$$

there would be in the simplified equation no term on the right corresponding to 2 on the left; so that, in equating the coefficients of like powers of x, 2, which is $2x^0$, would have to be put equal to $0x^0$; that is, $2 = 0$, an absurdity.

EXERCISE 131.

Expand to four terms:

1. $\frac{1}{1+2x}$.
2. $\frac{1}{2-3x}$.
3. $\frac{1+x}{2+3x}$.
4. $\frac{1-x}{1+x+x^2}$.
5. $\frac{5-2x}{1+x-x^2}$.
6. $\frac{2-3x}{1-2x+3x^2}$.
7. $\frac{3+x}{1-x-x^2}$.
8. $\frac{1+x}{1+x+x^2}$.
9. $\frac{1-8x}{1-x-6x^2}$.

Expand to five terms:

10. $\frac{4}{2+x}$.
11. $\frac{2-x}{3+x}$.
12. $\frac{5-2x}{1+3x-x^2}$.
13. $\frac{x^2-x+1}{x(x-2)}$.
14. $\frac{3x-2}{x(x-1)^2}$.
15. $\frac{x^2-x+1}{(x-1)(x^2+1)}$.

Partial Fractions.

398. To resolve a fraction into *partial fractions* is to express it as the sum of a number of fractions of which the respective denominators are the factors of the denominator of the given fraction. This is the reverse of the process of *adding* fractions that have different denominators.

Resolution into partial fractions may be easily accomplished by the use of **indeterminate coefficients** and the theorem of § 396.

In decomposing a given fraction into its simplest partial fractions, it is important to determine what form the assumed fractions must have. Since the given fraction is the *sum* of the required partial fractions, each assumed denominator must be a factor of the given denominator; moreover, all the factors of the given denominator must be taken as denominators of the assumed fractions.

Since the required partial fractions are to be in their simplest form, incapable of further decomposition, the numerator of each required fraction must be assumed with reference to this condition. Thus, if the denominator is x^n or $(x \pm a)^n$, the assumed fraction must be of the form $\frac{A}{x^n}$ or $\frac{A}{(x \pm a)^n}$; for, if it had the form $\frac{Ax+B}{x^n}$ or $\frac{Ax+B}{(x \pm a)^n}$, it could be decomposed into two fractions, and the partial fractions would not be in the simplest form possible.

When all the monomial factors, and all the binomial factors, of the form $x \pm a$, have been removed from the denominator of the given expression, there may remain quadratic factors that cannot be further resolved; and the numerators corresponding to these quadratic factors may each contain the first power of x, so that the assumed fractions must have either the form $\frac{Ax+B}{x^2 \pm ax + b}$ or the form $\frac{Ax+B}{x^2+b}$.

1. Resolve $\frac{3}{x^3+1}$ into partial fractions.

Since $x^3 + 1 = (x+1)(x^2 - x + 1)$, the denominators will be $x+1$ and $x^2 - x + 1$.

Assume $$\frac{3}{x^3+1} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1};$$

then, $$3 = A(x^2-x+1) + (Bx+C)(x+1)$$
$$= (A+B)x^2 + (B+C-A)x + (A+C);$$

whence, $$3 = A + C,\ B + C - A = 0,\ A + B = 0,$$

and $$A = 1,\ B = -1,\ C = 2.$$

Therefore, $$\frac{3}{x^3+1} = \frac{1}{x+1} - \frac{x-2}{x^2-x+1}.$$

2. Resolve $\dfrac{4x^3 - x^2 - 3x - 2}{x^2(x+1)^2}$ into partial fractions.

The denominators may be x, x^2, $x+1$, $(x+1)^2$.

Assume $\dfrac{4x^3 - x^2 - 3x - 2}{x^2(x+1)^2} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x+1} + \dfrac{D}{(x+1)^2}$.

$$\therefore 4x^3 - x^2 - 3x - 2 = Ax(x+1)^2 + B(x+1)^2 + Cx^2(x+1) + Dx^2$$
$$= (A+C)x^3 + (2A+B+C+D)x^2 + (A+2B)x + B;$$

whence,
$$A + C = 4,$$
$$2A + B + C + D = -1,$$
$$A + 2B = -3,$$
$$B = -2;$$

and
$$\therefore B = -2,\ A = 1,\ C = 3,\ D = -4.$$

Therefore, $\dfrac{4x^3 - x^2 - 3x - 2}{x^2(x+1)^2} = \dfrac{1}{x} - \dfrac{2}{x^2} + \dfrac{3}{x+1} - \dfrac{4}{(x+1)^2}$.

EXERCISE 132.

Resolve into partial fractions:

1. $\dfrac{7x+1}{(x+4)(x-5)}$.

2. $\dfrac{7x-1}{(1-2x)(1-3x)}$.

3. $\dfrac{5x-1}{(2x-1)(x-5)}$.

4. $\dfrac{x-2}{(x-5)(x+2)}$.

5. $\dfrac{3}{x^3-1}$.

6. $\dfrac{x^2 - x - 3}{x(x^2-4)}$.

7. $\dfrac{3x^2-4}{x^2(x+5)}$.

8. $\dfrac{7x^2-x}{(x-1)^2(x+2)}$.

9. $\dfrac{2x^2-7x+1}{x^3-1}$.

10. $\dfrac{7x-1}{(6x+1)(x-1)}$.

11. $\dfrac{x^2-3}{(x^2+1)(x+2)}$.

12. $\dfrac{x^2-x+1}{(x^2+1)(x-1)^2}$.

CHAPTER XXV.

BINOMIAL THEOREM.

399. Binomial Theorem, Positive Integral Exponent. By successive multiplication we obtain the following identities:

$$(a+b)^2 = a^2 + 2ab + b^2;$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3;$$

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

The expressions on the right may be written in a form better adapted to show the law of their formation:

$$(a+b)^2 = a^2 + 2ab + \frac{2\cdot 1}{1\cdot 2}b^2;$$

$$(a+b)^3 = a^3 + 3a^2b + \frac{3\cdot 2}{1\cdot 2}ab^2 + \frac{3\cdot 2\cdot 1}{1\cdot 2\cdot 3}b^3;$$

$$(a+b)^4 = a^4 + 4a^3b + \frac{4\cdot 3}{1\cdot 2}a^2b^2 + \frac{4\cdot 3\cdot 2}{1\cdot 2\cdot 3}ab^3 + \frac{4\cdot 3\cdot 2\cdot 1}{1\cdot 2\cdot 3\cdot 4}b^4.$$

NOTE. The dot between the Arabic figures means the same as the sign ×.

400. Let n represent the exponent of $(a+b)$ in any one of these identities; then, in the expressions on the right, we observe that the following laws hold true:

1. The number of terms is $n+1$.

2. The first term is a^n, and the exponent of a is one less in each succeeding term.

3. The first power of b occurs in the second term, the second power in the third term, and the exponent of b is one greater in each succeeding term.

4. The sum of the exponents of a and b in any term is n.

5. The coefficients of the terms taken in order are 1; n; $\frac{n(n-1)}{1 \cdot 2}$; $\frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}$; and so on.

401. Consider the coefficient of any term; the number of factors in the numerator is the same as the number of factors in the denominator, and the number of factors in each is the same as the exponent of b in that term; this exponent is one less than the number of the term.

402. Proof of the Theorem. That the laws of § 400 hold true when the exponent is *any* positive integer is shown as follows:

We know that the laws hold for the fourth power; suppose, for the moment, that they hold for the kth power. We shall then have

$$(a+b)^k = a^k + ka^{k-1}b + \frac{k(k-1)}{1 \cdot 2} a^{k-2}b^2 + \frac{k(k-1)(k-2)}{1 \cdot 2 \cdot 3} a^{k-3}b^3 + \dots\dots \quad (1)$$

Multiply both members of (1) by $a+b$; the result is

$$(a+b)^{k+1} = a^{k+1} + (k+1)a^k b + \frac{(k+1)k}{1 \cdot 2} a^{k-1}b^2 + \frac{(k+1)k(k-1)}{1 \cdot 2 \cdot 3} a^{k-2}b^3 + \dots\dots \quad (2)$$

In (1) put $k+1$ for k; this gives

$$\begin{aligned}(a+b)^{k+1} &= a^{k+1} + (k+1)a^k b + \frac{(k+1)(k+1-1)}{1 \cdot 2} a^{k-1}b^2 \\ &\quad + \frac{(k+1)(k+1-1)(k+1-2)}{1 \cdot 2 \cdot 3} a^{k-2}b^3 + \dots\dots \\ &= a^{k+1} + (k+1)a^k b + \frac{(k+1)k}{1 \cdot 2} a^{k-1}b^2 \\ &\quad + \frac{(k+1)k(k-1)}{1 \cdot 2 \cdot 3} a^{k-2}b^3 + \dots\dots \qquad (3)\end{aligned}$$

Equation (3) is seen to be the same as equation (2).

Hence, (1) holds when we put $k+1$ for k; that is, if the laws of § 400 hold for the kth power, they must hold for the $(k+1)$th power.

But the laws hold for the fourth power; therefore, they must hold for the fifth power.

Holding for the fifth power, they must hold for the sixth power; and so on for any positive integral power.

Therefore, they must hold for the nth power, if n is a positive integer; and we have

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1 \cdot 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}a^{n-3}b^3 + \cdots\cdots \qquad \text{(A)}$$

NOTE. The above proof is an example of a proof by *mathematical induction.* See § 134.

403. This formula is known as the **binomial theorem.**

The expression on the right is known as the **expansion** of $(a+b)^n$; this expansion is a *finite series* when n is a positive integer. That the series is finite may be seen as follows:

In writing out the successive coefficients we shall finally arrive at a coefficient that contains the factor $n-n$; and, therefore, this term will vanish. The coefficients of all the succeeding terms likewise contain the factor $n-n$, and, therefore, all these terms will vanish.

404. If a and b are interchanged, the identity (A) may be written

$$(a+b)^n = (b+a)^n = b^n + nb^{n-1}a + \frac{n(n-1)}{1 \cdot 2}b^{n-2}a^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}b^{n-3}a^3 + \cdots\cdots$$

This last expansion is the expansion of (A) written in reverse order. Comparing the two expansions, we see that the coefficient of the last term is the same as the coefficient of the first term; the coefficient of the last term but one is the same as the coefficient of the first term but one; and so on.

In general, the coefficient of the rth term from the end is the same as the coefficient of the rth term from the beginning. In writing an expansion by the binomial theorem, after arriving at the middle term, we can shorten the work by observing that the remaining coefficients are those already found, taken in reverse order.

405. If b is negative, the terms that involve even powers of b will be positive, and the terms that involve odd powers of b will be negative. Hence,

$$(a-b)^n = a^n - na^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 - \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3 + \cdots\cdots \qquad \text{(B)}$$

Also, putting 1 for a and x for b in (A) and (B).

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1\cdot 2}x^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3 + \cdots\cdots \qquad \text{(C)}$$

$$(1-x)^n = 1 - nx + \frac{n(n-1)}{1\cdot 2}x^2 - \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3 + \cdots\cdots \qquad \text{(D)}$$

1. Expand $(1 + 2x)^5$.

In (C) put $2x$ for x and 5 for n. The result is

$$(1 + 2x)^5 = 1 + 5(2x) + \frac{5 \cdot 4}{1 \cdot 2} 4x^2 + \frac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3} 8x^3$$
$$+ \frac{5 \cdot 4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3 \cdot 4} 16x^4 + \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} 32x^5$$
$$= 1 + 10x + 40x^2 + 80x^3 + 80x^4 + 32x^5.$$

2. Expand to three terms $\left(\frac{1}{x} - \frac{2x^2}{3}\right)^6$.

Put a for $\frac{1}{x}$, and b for $\frac{2x^2}{3}$; then, by (B),

$$(a - b)^6 = a^6 - 6a^5b + 15a^4b^2 - \cdots\cdots$$

Replace a and b by their values,

$$\left(\frac{1}{x} - \frac{2x^2}{3}\right)^6 = \left(\frac{1}{x}\right)^6 - 6\left(\frac{1}{x}\right)^5\left(\frac{2x^2}{3}\right) + 15\left(\frac{1}{x}\right)^4\left(\frac{2x^2}{3}\right)^2 - \cdots\cdots$$
$$= \frac{1}{x^6} - \frac{4}{x^3} + \frac{20}{3} - \cdots\cdots$$

406. Any Required Term. From (A) it is evident (§ 400) that the $(r + 1)$th term in the expansion of $(a + b)^n$ is

$$\frac{n(n-1)(n-2) \cdots\cdots \text{ to } r \text{ factors}}{1 \times 2 \times 3 \cdots\cdots r} a^{n-r}b^r.$$

NOTE. In finding the coefficient of the $(r + 1)$th term, write the series of factors $1 \times 2 \times 3 \cdots\cdots r$ for the denominator of the coefficient, then write over this series the factors $n(n-1)(n-2)$, etc., writing just as many factors in the numerator as there are in the denominator.

The $(r + 1)$th term in the expansion of $(a - b)^n$ is the same as the above if r is even, and the negative of the above if r is odd.

Find the eighth term of $\left(4-\frac{x^2}{2}\right)^{10}$.

Here $a=4,\ b=\frac{x^2}{2},\ n=10,\ r=7.$

The term required is $\frac{10\cdot9\cdot8\cdot7\ 6\cdot5\cdot4}{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7}(4)^3\left(-\frac{x^2}{2}\right)^7$,

which reduces to $-60x^{14}$.

407. A trinomial may be expanded by the binomial theorem as follows:

Expand $(1+2x-x^2)^3$.

Put $2x-x^2=z$.

Then, $(1+z)^3=1+3z+3z^2+z^3$.

Replace z by $2x-x^2$.

$$\begin{aligned}\therefore (1+2x-x^2)^3 &= 1+3(2x-x^2)+3(2x-x^2)^2+(2x-x^2)^3\\ &= 1+6x+9x^2-4x^3-9x^4+6x^5-x^6.\end{aligned}$$

Exercise 133.

Expand:

1. $(a+b)^7$.
2. $(x-2)^5$.
3. $(3x-2y)^4$.
4. $\left(\frac{x}{y}-\frac{y}{x}\right)^6$.
5. $(4+3y)^4$.
6. $(a^3+b)^7$.
7. $(m^2+n^3)^8$.
8. $(a-b^3)^7$.
9. $(a^{\frac{1}{2}}+b^{\frac{2}{3}})^5$.
10. $(a^{-1}+b^{-2})^3$.
11. $(m^{-\frac{1}{2}}+n^2)^4$.
12. $(x^{-2}+z^{\frac{2}{3}})^6$.
13. $(2x^2+y^{\frac{1}{2}})^5$.
14. $(a^{\frac{1}{3}}-c^{\frac{2}{3}})^4$.
15. $(2a^2-\frac{1}{2}\sqrt{a})^5$.
16. $\left(\frac{a^2}{b}-\frac{\sqrt{b}}{2a}\right)^7$.
17. $\left(\frac{2a}{b^2}+\frac{1}{2}b\sqrt{a}\right)^8$.
18. $(2x^2y^{-1}-y\sqrt{y})^4$.
19. $\left(\frac{a\sqrt[3]{a}}{\sqrt[7]{b^5}}+\frac{\sqrt[3]{b}}{a}\right)^7$.
20. $\left(\frac{\sqrt{a}}{\sqrt[3]{b^2}}-\frac{1}{2}\sqrt{\frac{b}{a}}\right)^5$.
21. $(2ab^{-2}-ba^{\frac{1}{2}})^7$.

22. $\left(\frac{a}{b}\sqrt{\frac{c}{d}} - \frac{c}{d}\right)^6.$ 23. $\left(\frac{a}{b}\sqrt{\frac{b}{a}} - \frac{b^{\frac{1}{2}}}{\sqrt{ac}}\right)^5.$

24. Find the fourth term of $(2x - 3y)^7$.

25. Find the ninety-seventh term of $(2a - b)^{100}$.

Note. As the expansion has 101 terms, the ninety-seventh term from the beginning is the fifth term from the end.

26. Find the eighth term of $(3x - y)^{11}$.

27. Find the tenth term of $(2a^2 - \frac{1}{2}a)^{20}$.

28. Find the fifth term of $(a - 2\sqrt{b})^{25}$.

29. Find the eleventh term of $(2 - a)^{16}$.

30. Find the fifteenth term of $(x + y)^{20}$.

31. Find the fourth term of $(3 - 2x^2)^9$.

32. Find the twelfth term of $(a^2 - a\sqrt{x})^{17}$.

33. Find the seventh term of $(y^2 - 1)^{38}$.

34. Find the fifth term of $(\frac{1}{2}a - b\sqrt{b})^{21}$.

35. Find the fourth term of $(\sqrt{a} - \sqrt[3]{b^2})^{20}$.

36. Find the third term of $(\sqrt{a} - \sqrt{-b})^7$.

37. Find the sixth term of $(\sqrt[3]{a^2} - \sqrt{-1})^9$.

38. Find the eighth term of $(\sqrt{\frac{2}{3}a} + \sqrt{\frac{3}{4}x})^{20}$.

39. Find the ninth term of $(x\sqrt{-1} + y\sqrt{-1})^{16}$.

40. Find the fifth term of $\left(a^3b - \frac{3b^{-2}}{\sqrt{a^5}}\right)^{81}$.

41. Find the seventh term of $(x + x^{-1})^{2n}$.

408. Binomial Theorem, Any Exponent. We have seen (§ 402) that when n is a positive integer we have the identity

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1\cdot 2}x^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3 + \cdots\cdots$$

We proceed to the case of fractional and negative exponents.

I. Suppose n is a positive fraction, $\frac{p}{q}$, in which p and q are positive integers. We may assume that

$$(1+x)^{\frac{p}{q}} = (A + Bx + Cx^2 + Dx^3 + \cdots\cdots)^q, \qquad (1)$$

provided x is so taken that the series

$$A + Bx + Cx^2 + Dx^3 + \cdots\cdots$$

is convergent, § 392.

That this assumption is allowable may be seen as follows: Expand both members of (1). We obtain

$$1 + px + \frac{p(p-1)}{1\cdot 2}x^2 + \frac{p(p-1)(p-2)}{1\cdot 2\cdot 3}x^3 + \cdots\cdots,$$

and $A^q + qA^{q-1}Bx + \left[\frac{q(q-1)}{1\cdot 2}A^{q-2}B^2 + qA^{q-1}C\right]x^2 + \cdots\cdots$

In the first k coefficients of the second series there enter only the first k of the coefficients $A, B, C, D, \cdots\cdots$ If, then, we equate the coefficients of corresponding terms in the two series (§ 396) as far as the kth term, *we shall have just k equations to find k unknown numbers $A, B, C, D, \cdots\cdots$* Hence, the assumption made in (1) is allowable.

Equating the two first terms and the two second terms, we obtain

$$A^q = 1, \qquad \therefore A = 1;$$

$$qA^{q-1}B = p, \text{ or } qB = p, \qquad \therefore B = \frac{p}{q}.$$

Extracting the qth root of both members of (1), we have

$$(1+x)^{\frac{p}{q}} = 1 + \frac{p}{q}x + Cx^2 + Dx^3 + \cdots\cdots, \tag{2}$$

where x is to be so taken that the series on the right is convergent.

II. Suppose n is a negative number, integral or fractional. Let $n = -m$, so that m is positive; then,

$$(1+x)^n = (1+x)^{-m} = \frac{1}{(1+x)^m}.$$

From (2), whether m is integral or fractional, we may assume

$$\frac{1}{(1+x)^m} = \frac{1}{1 + mx + cx^2 + dx^3 + \cdots\cdots}.$$

By actual division this gives an equation in the form

$$(1+x)^{-m} = 1 - mx + Cx^2 + Dx^3 + \cdots\cdots \tag{3}$$

409. It appears from (2) and (3) (§ 408) that whether n is integral or fractional, positive or negative, we may assume

$$(1+x)^n = 1 + nx + Cx^2 + Dx^3 + \cdots\cdots,$$

provided the series on the right is convergent.

Square both members,

$$\begin{aligned}(1+2x+x^2)^n = 1 + 2nx &+ 2Cx^2 + 2Dx^3 + \cdots\cdots \\ &+ n^2x^2 \;\; + 2nCx^3 + \cdots\cdots\end{aligned} \tag{1}$$

Also, since

$$(1+y)^n = 1 + ny + Cy^2 + Dy^3 + \cdots\cdots,$$

we have, putting $2x + x^2$ for y,

$$\begin{aligned}(1+2x+x^2)^n &= 1 + n(2x+x^2) + C(2x+x^2)^2 \\ &\qquad + D(2x+x^2)^3 \cdots\cdots \\ &= 1 + 2nx + nx^2 \;\; + 4Cx^3 + \cdots\cdots \\ &\qquad\qquad + 4Cx^2 + 8Dx^3 + \cdots\cdots\end{aligned} \tag{2}$$

Equate corresponding coefficients in (1) and (2),

$$n + 4C = 2C + n^2,$$
$$4C + 8D = 2D + 2nC.$$
$$\therefore 2C = n^2 - n, \text{ and } \quad C = \frac{n(n-1)}{1 \cdot 2};$$
$$3D = (n-2)C, \text{ and } D = \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3};$$

and so on.

Hence, whether n is integral or fractional, positive or negative, we have

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1 \cdot 2}x^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}x^3 + \cdots\cdots,$$

provided, always, x is so taken that the series on the right is convergent.

The series obtained will be an infinite series unless n is a positive integer, § 403.

410. If x is negative,

$$(1-x)^n = 1 - nx + \frac{n(n-1)}{1 \cdot 2}x^2 - \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}x^3 + \cdots\cdot$$

Also, if $x < a$,

$$\begin{aligned}(a+x)^n &= a^n\left(1 + \frac{x}{a}\right)^n \\ &= a^n\left[1 + n\frac{x}{a} + \frac{n(n-1)}{1 \cdot 2}\frac{x^2}{a^2} + \cdots\cdots\right] \\ &= a^n + na^{n-1}x + \frac{n(n-1)}{1 \cdot 2}a^{n-2}x^2 + \cdots\cdots;\end{aligned}$$

if $x > a$,

$$\begin{aligned}(a+x)^n = (x+a)^n &= x^n\left(1 + \frac{a}{x}\right)^n \\ &= x^n\left[1 + n\frac{a}{x} + \frac{n(n-1)}{1 \cdot 2}\frac{a^2}{x^2} + \cdots\cdots\right] \\ &= x^n + nax^{n-1} + \frac{n(n-1)}{1 \cdot 2}a^2x^{n-2} + \cdots\cdots\end{aligned}$$

1. Expand $(1+x)^{\frac{1}{3}}$.

$$(1+x)^{\frac{1}{3}} = 1 + \tfrac{1}{3}x + \frac{\frac{1}{3}(\frac{1}{3}-1)}{1\cdot 2}x^2 + \frac{\frac{1}{3}(\frac{1}{3}-1)(\frac{1}{3}-2)}{1\cdot 2\cdot 3}x^3 + \cdots\cdots$$

$$= 1 + \tfrac{1}{3}x - \frac{2}{3\cdot 6}x^2 + \frac{2\cdot 5}{3\cdot 6\cdot 9}x^3 - \cdots\cdots$$

The above equation is true only for those values of x that make the series convergent.

2. Expand $\dfrac{1}{\sqrt[4]{1-x}}$.

$$\frac{1}{\sqrt[4]{1-x}} = (1-x)^{-\frac{1}{4}}$$

$$= 1 - (-\tfrac{1}{4})x + \frac{-\frac{1}{4}\cdot -\frac{5}{4}}{1\cdot 2}x^2 - \frac{-\frac{1}{4}\cdot -\frac{5}{4}\cdot -\frac{9}{4}}{1\cdot 2\cdot 3}x^3 + \cdots\cdots$$

$$= 1 + \tfrac{1}{4}x + \frac{1\cdot 5}{4\cdot 8}x^2 + \frac{1\cdot 5\cdot 9}{4\cdot 8\cdot 12}x^3 + \cdots\cdots,$$

if x is so taken that the series is convergent.

A root may often be extracted by means of an expansion.

3. Extract the cube root of 344 to six decimal places.

$$344 = 343\left(1 + \frac{1}{343}\right) = 7^3\left(1 + \frac{1}{343}\right).$$

$$\therefore \sqrt[3]{344} = 7\left(1 + \frac{1}{343}\right)^{\frac{1}{3}}$$

$$= 7\left[1 + \frac{1}{3}\left(\frac{1}{343}\right) + \frac{\frac{1}{3}(\frac{1}{3}-1)}{1\cdot 2}\left(\frac{1}{343}\right)^2 + \cdots\cdots\right]$$

$$= 7\,(1 + 0.000971817 - 0.000000944 + \cdots\cdots)$$

$$= 7.006796.$$

4. Find the eighth term of $\left(x - \dfrac{3}{4\sqrt{x}}\right)^{-\frac{1}{2}}$.

Here $\qquad a = x,\ b = \dfrac{3}{4\sqrt{x}} = \dfrac{3}{4x^{\frac{1}{2}}},\ n = -\dfrac{1}{2},\ r = 7.$

The term is $\dfrac{-\frac{1}{2}\cdot -\frac{3}{2}\cdot -\frac{5}{2}\cdot -\frac{7}{2}\cdot -\frac{9}{2}\cdot -\frac{11}{2}\cdot -\frac{13}{2}}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}x^{-\frac{15}{2}}\left(-\dfrac{3}{4x^{\frac{1}{2}}}\right)^7,$

or $$\frac{1\cdot 3\cdot 5\cdot 7\cdot 9\cdot 11\cdot 13\cdot 3^7}{2\cdot 4\cdot 6\cdot 8\cdot 10\cdot 12\cdot 14\cdot 4^7\cdot x^{11}}.$$

EXERCISE 134.

Expand to four terms:

1. $(1+x)^{\frac{1}{2}}$.

2. $(1+x)^{-2}$.

3. $(1+x)^{-\frac{1}{2}}$.

4. $(1-x)^{\frac{2}{3}}$.

5. $(1-x)^{-\frac{2}{3}}$.

6. $(1+x)^{\frac{5}{3}}$.

7. $(1+x)^{-\frac{5}{3}}$.

3. $(1+x)^{-3}$.

9. $(1+5x)^{-5}$.

10. $(1+5x)^{\frac{5}{3}}$.

11. $(2x+3y)^{\frac{3}{4}}$.

12. $(2x+3y)^{-\frac{3}{4}}$.

13. $\dfrac{1}{\sqrt[4]{a^2-x}}$.

14. $\dfrac{1}{\sqrt[5]{(a-x)^2}}$.

15. Find the fourth term of $\left(a-\dfrac{3}{2\sqrt{x}}\right)^{\frac{1}{2}}$.

16. Find the fifth term of $\dfrac{1}{\sqrt[3]{(a-2x)^2}}$.

17. Find the third term of $(4-7x)^{\frac{2}{7}}$.

18. Find the sixth term of $(a^2-2ax)^{\frac{5}{3}}$.

19. Find the fifth term of $(1-2x)^{-\frac{1}{2}}$.

20. Find the fifth term of $(1-x)^{-3}$.

21. Find the seventh term of $(1-x)^{\frac{1}{2}}$.

22. Find the third term of $(1+x)^{-\frac{1}{2n}}$.

23. Find the fourth term of $(1+x)^{-\frac{5}{3}}$.

24. Find the sixth term of $\left(2-\dfrac{1}{x}\right)^{-\frac{2}{7}}$.

25. Find the fifth term of $(2x-3y)^{-\frac{3}{4}}$.

26. Find the fourth term of $(1-5x)^{-\frac{5}{3}}$.

EXERCISE 135. — GENERAL REVIEW.

1. Add $(a-b)x^2+(b-c)y^2+(c-a)z^2$; $(b-c)x^2+(c-a)y^2+(a-b)z^2$; $(c-a)x^2+(a-b)y^2+(b-c)z^2$.

2. Add $(a+b)x+(b+c)y-(c+a)z$; $(b+c)z+(c+a)x-(a+b)y$; $(a+c)y+(a+b)z-(b+c)x$.

3. From $4x^3-6x^2+8x-7$ take the sum of $8x^3+7-8x^2+7x$ and $-9x^3-8x^2+4x+4$.

4. Find the product of $a^p-3a^{p-1}+4a^{p-2}-6a^{p-3}+5a^{p-4}$ and $2a^3-a^2+a$.

5. Divide $1-6x^5+5x^6$ by $1-2x+x^2$.

6. Divide $4h^{x+1}-30h^x+19h^{x-1}+5h^{x-2}+9h^{x-4}$ by $h^{x-3}-7h^{x-4}+2h^{x-5}-3h^{x-6}$.

7. Simplify $3\{a-2(b-c)\}-[4b+\{2b-(c-a)\}]$.

8. Find the factors of $10x^2+79x-8$.

9. Find the H. C. F. of $2x^3+x^2+4x-7$ and x^3-2x^2+1.

10. Reduce to lowest terms $\dfrac{a^4+9a-20}{5a^4+9a^3-64}$.

11. If $a=4$, $b=\frac{1}{2}$, $c=-2$, find the numerical value of
$$3a^2+\frac{2ab^2}{c}-\frac{c^3}{b^2}.$$

12. If $x=\dfrac{a+b}{2}$, show that $\left(\dfrac{x-a}{x-b}\right)^3-\dfrac{x-2a+b}{x+a-2b}=0$.

13. If $x=2y+3z$, show that
$$x^3-8y^3-27z^3-18xyz=0.$$

14. Resolve into factors $6x^2+5x-4$.

15. Resolve into factors $x^4+2x^3-13x^2-38x-24$.

16. Resolve into factors $12x^3+20x^2-x-6$.

17. A boy bought a number of apples at the rate of 5 for 2 cents. He sold half of them at the rate of 2 for a cent and the rest at the rate of 3 for a cent, and cleared a cent by the transaction. How many did he buy?

Find the H. C. F. and the L. C. M. of:

18. $3x^5 - 5x^3 + 2$, and $2x^5 - 5x^2 + 3$.

19. $3x^3 + 10x^2 + 7x - 2$, and $3x^3 + 13x^2 + 17x + 6$.

20. $4x^4 - 9x^2 + 6x - 1$, and $6x^3 - 7x^2 + 1$.

21. $x^5 + 11x - 12$, and $x^5 + 11x^3 + 54$.

22. $2x^3 + 5x^2y - 5xy^2 + y^3$, and $2x^3 - 7x^2y + 5xy^2 - y^3$.

Simplify:

23. $\dfrac{x+1}{x(x-2)} - \dfrac{3x+2}{x(x+1)} + \dfrac{2x-1}{x^2-x-2}.$

24. $\dfrac{1+x}{1-x} + \dfrac{1-x}{1+x} - \dfrac{1+x^2}{1-x^2} - \dfrac{1-x^2}{1+x^2}.$

25. $\dfrac{x-1}{(x+2)(x+5)} - \dfrac{2(x+2)}{(x+5)(x-1)} + \dfrac{x+5}{(x-1)(x+2)}.$

26. $\dfrac{1}{ax-a^2} + \dfrac{1}{ax+2a^2} - \dfrac{3}{x^2+ax-2a^2}.$

27. $\dfrac{x}{(x+3)(x-1)} + \dfrac{x-1}{(x+3)(2-x)} - \dfrac{x-3}{(2-x)(1-x)}$

28. $\dfrac{\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}}{\dfrac{1}{a^2}+\dfrac{1}{b^2}-\dfrac{1}{c^2}+\dfrac{2}{ab}}.$

30. $\dfrac{1-\dfrac{1-\frac{1}{3}(1-x)}{2}}{1-\dfrac{1-\frac{1}{2}(1-x)}{3}}.$

29. $\dfrac{\dfrac{1}{1-a}-\dfrac{1}{1+a}}{\dfrac{a}{1-a}+\dfrac{1}{1+a}}.$

31. $\dfrac{\dfrac{1}{a}+\dfrac{1}{ab^3}}{b-1+\dfrac{1}{b}}.$

32. $$\frac{1}{x-1+\dfrac{1}{1+\dfrac{x}{4-x}}}.$$

33. $$\frac{1}{1+\dfrac{a}{1+a+\dfrac{2a^2}{1+a}}}.$$

34. Two passengers have together 400 pounds of baggage. One pays $1.20, the other $1.80, for excess above the weight allowed. If all the baggage had belonged to one person, he would have had to pay $4.50. How much baggage is allowed free?

Solve:

35. $$\frac{6x+13}{15}-\frac{9x+15}{5x-25}+3=\frac{2x+15}{5}.$$

36. $$\frac{2x+a}{3(x-a)}+\frac{3x-a}{2(x+a)}=2\tfrac{1}{6}.$$

37. $$\left.\begin{aligned}\frac{x}{3}+\frac{5}{y}&=4\tfrac{1}{3}\\ \frac{x}{6}+\frac{10}{y}&=2\tfrac{2}{3}\end{aligned}\right\}$$

38. $$\left.\begin{aligned}\frac{5x}{0.7}+\frac{0.3}{y}&=6\\ \frac{10x}{7}+\frac{9}{y}&=31\end{aligned}\right\}$$

39. $$\left.\begin{aligned}\frac{2x}{a}+\frac{3y}{b}-\frac{4z}{c}&=1\\ \frac{4x}{a}+\frac{2y}{b}-\frac{3z}{c}&=3\\ \frac{3x}{a}+\frac{4y}{b}-\frac{2z}{c}&=5\end{aligned}\right\}$$

40. $$\left.\begin{aligned}\frac{a}{x}+\frac{b}{y}+\frac{c}{z}&=3\\ \frac{a}{x}+\frac{b}{y}-\frac{c}{z}&=1\\ \frac{2a}{x}-\frac{b}{y}-\frac{c}{z}&=0\end{aligned}\right\}$$

Find the arithmetical value of:

41. $36^{\frac{1}{2}}$; $27^{\frac{1}{3}}$; $16^{\frac{1}{4}}$; $32^{\frac{1}{5}}$; $4^{\frac{5}{2}}$; $8^{\frac{2}{3}}$; $27^{\frac{5}{3}}$; $64^{\frac{2}{3}}$.

42. $32^{\frac{3}{5}}$; $64^{\frac{5}{6}}$; $81^{\frac{3}{4}}$; $(3\tfrac{3}{8})^{\frac{1}{3}}$; $(5\tfrac{1}{16})^{\frac{1}{4}}$; $(1\tfrac{9}{16})^{\frac{3}{2}}$.

43. $(0.25)^{\frac{1}{2}};\ (0.027)^{\frac{2}{3}};\ 49^{-\frac{3}{2}};\ 32^{-\frac{2}{5}};\ 81^{-\frac{3}{4}}.$

44. $36^{-\frac{1}{2}};\ 27^{-\frac{1}{3}};\ (\frac{9}{16})^{-\frac{3}{2}};\ (0.16)^{-\frac{3}{2}};\ (0.0016)^{-\frac{3}{4}}.$

Arrange in ascending order of magnitude:

45. $6\sqrt{7};\ 9\sqrt{3};\ 5\sqrt{10}.$ 46. $4\sqrt{6};\ 3\sqrt[3]{3};\ 5\sqrt[3]{2}.$

Simplify:

47. $\frac{2}{5}\sqrt{3} \times \frac{4}{9}\sqrt{5} \div \frac{6}{7}\sqrt{2}.$ 48. $\dfrac{2\sqrt{10}}{3\sqrt{27}} \times \dfrac{7\sqrt{48}}{5\sqrt{14}} \div \dfrac{4\sqrt{15}}{15\sqrt{21}}.$

49. $\dfrac{1}{2+\sqrt{3}}.$ 50. $\dfrac{1}{2-\sqrt{3}}.$ 51. $\dfrac{11}{2\sqrt{5}+3}.$ 52. $\dfrac{1}{2\sqrt{5}-3}.$

53. $\left\{(x^{5ab})^3 \times (x^{5b})^{-2}\right\}^{\frac{1}{3a-2}}.$ 54. $(x^{18a} \times x^{-12})^{\frac{1}{3a-2}}.$

55. $3(a^{\frac{1}{2}}+b^{\frac{1}{2}})^2 - 4(a^{\frac{1}{2}}+b^{\frac{1}{2}})(a^{\frac{1}{2}}-b^{\frac{1}{2}}) + (a^{\frac{1}{2}}-2b^{\frac{1}{2}})^2.$

56. $\left\{(a^m)^{m-\frac{1}{m}}\right\}^{\frac{1}{m+1}}.$ 58. $[\{(a^{-m})^{-n}\}^p]^q \div [\{(a^m)^n\}^{-p}]^{-q}$

57. $\left(\dfrac{x^{p+q}}{x^q}\right)^p \div \left(\dfrac{x^q}{x^{q-p}}\right)^{p-q}.$ 59. $\dfrac{x^{2p(q-1)} - y^{2q(p-1)}}{x^{p(q-1)} + y^{q(p-1)}}.$

60. $5\sqrt[3]{-320} - 2\sqrt[3]{-1715} + 3\frac{1}{3}\sqrt[3]{135}.$

61. $2\sqrt{18} - 3\sqrt{8} + 2\sqrt{50};\ \ \sqrt[3]{81} + \sqrt[3]{24} - \sqrt[3]{192}.$

62. $\frac{3}{2}\sqrt{\frac{5}{9}} + \sqrt{80} - \frac{1}{4}\sqrt{20};\ \ 8\sqrt{\frac{5}{16}} + 10\sqrt{\frac{4}{5}} - 2\sqrt{\frac{5}{4}}.$

Extract the square root of:

63. $9x^{-4} - 18x^{-3}y^{\frac{1}{2}} + 15x^{-2}y - 6x^{-1}y^{\frac{3}{2}} + y^2.$

Extract the cube root of:

64. $8x^3 + 12x^2 - 30x - 35 + 45x^{-1} + 27x^{-2} - 27x^{-3}.$

Resolve into prime factors with fractional exponents:

65. $\sqrt[3]{12}$, $\sqrt[4]{72}$, $\sqrt[6]{96}$, $\sqrt[8]{64}$; and find their product.

Solve:

66. $\dfrac{5}{x-2}-\dfrac{4}{x}=\dfrac{3}{x+6}.$

67. $\dfrac{x+3}{2x-7}-\dfrac{2x-1}{x-3}=0.$

68. $\dfrac{x+4}{x-4}+\dfrac{x-2}{x-3}=6\frac{1}{3}.$

69. $\dfrac{x-a}{x-b}+\dfrac{x-b}{x-a}=\dfrac{a^2+b^2}{ab}.$

70. $\dfrac{ax+b}{a+bx}=\dfrac{cx+d}{c+dx}.$

71. $ax^2-\dfrac{6c^2}{a+b}=cx-bx^2.$

72. $\left.\begin{aligned}3xy-5y^2&=1\\5xy+3x^2&=22\end{aligned}\right\}$

73. $\left.\begin{aligned}x^2+10xy&=11\\5xy-3y^2&=2\end{aligned}\right\}$

74. $\left.\begin{aligned}&\sqrt{x+y}=\sqrt{y}+2\\&x-y=7\end{aligned}\right\}$

75. $\left.\begin{aligned}&x^2+xy+y^2=52\\&xy-x^2=8\end{aligned}\right\}$

Form the equation of which the roots are:

76. $a-b,\ a+b.$

77. $a-2b,\ a+3b.$

78. $a+2b,\ 2a+b.$

79. $1+\sqrt{3},\ 1-\sqrt{3}.$

80. $-1+\sqrt{3},\ -1-\sqrt{3}.$

81. $1+\sqrt{-3},\ 1-\sqrt{-3}.$

82. A vessel that has two pipes can be filled in 2 hours less time by one than by the other, and by both together in 1 hour 52 minutes 30 seconds. How long will it take each pipe alone to fill the vessel?

83. A number is expressed by two digits, the second of which is the square of the other, and when 54 is added its digits are interchanged. Find the number.

84. Divide 35 into two parts such that the sum of the two fractions formed by dividing each part by the other may be $2\frac{1}{12}$.

85. A number consists of three digits in arithmetical progression; and this number divided by the sum of its digits is equal to 26; but if 198 is added to the number, the digits in the units' and hundreds' places will be interchanged. Find the number.

86. The sum of the squares of the extremes of four numbers in arithmetical progression is 200, and the sum of the squares of the means is 136. What are the numbers?

87. Show that if any even number of terms of the series 1, 3, 5 ..... is taken, the sum of the first half is to the sum of the second half in the ratio 1 : 3.

88. If a horse took 1 second more for each rod, he would travel $1\frac{7}{8}$ miles less per hour. Find his rate of traveling.

Solve:

89. $\left.\begin{aligned} x^3 + y^3 &= 18\sqrt{2} \\ x + y &= 3\sqrt{2} \end{aligned}\right\}$

90. $\left.\begin{aligned} 2x^2 + 3xy &= 8 \\ y^2 - 2xy &= 20 \end{aligned}\right\}$

Expand:

91. $(x - x^{-1})^3$.

92. $(\sqrt[3]{x} - \frac{1}{2}x)^4$.

93. $(2a^{\frac{1}{2}} - a^{-\frac{1}{2}})^3$.

94. $(2x^{-2} + x^{\frac{1}{2}})^5$.

95. $(\frac{1}{2}\sqrt{x} - \frac{1}{3}\sqrt[3]{x})^4$.

96. $(\frac{1}{3}\sqrt{x^{-3}} + \frac{1}{4}x^{-1})^3$.

97. If a, b, c, d are in continued proportion, prove that $b + c$ is a mean proportional between $a + b$ and $c + d$.

98. If $a + b : b + c = c + d : d + a$,

prove that $a = c$, or $a + b + c + d = 0$.

99. The number of eggs which can be bought for 1 dollar is equal to twice the number of cents which 32 eggs cost. How many eggs can be bought for 1 dollar?

100. Find two fractions whose sum is $\frac{7}{12}$, and whose difference is equal to their product.

101. The velocity of a falling body varies as the time during which it has fallen from rest, and the velocity at the end of 2 seconds is 64 ft. Find the velocity at the end of 6 seconds.

102. The distance through which a body falls from rest varies as the square of the time it falls; and a body falls 144 ft. in 3 seconds. How far does it fall in 4 seconds?

103. The volume of a gas varies directly as the absolute temperature and inversely as the pressure. If the volume of a gas is 1 cubic foot, when the pressure is 15 and the temperature 280, what will be the volume when the pressure is 35 and the temperature 320?

104. The difference between the first and second of four numbers in geometrical progression is 96; the difference between the third and fourth is 6. Find the numbers.

105. If a^2, b^2, c^2 are in arithmetical progression, prove that $b + c$, $c + a$, $a + b$ are in harmonical progression.

When $x = \infty$, and when $x = 0$, find the limit of:

106. $\frac{(2x - 3)(3 - 5x)}{7x^2 - 6x + 4}$.

107. $\frac{x^2 - x + 1}{(x^2 + 1)(x - 1)^2}$.

Resolve into factors and find all the values of x:

108. $x^4 - 5x^2 + 4 = 0$.

109. $x^6 - 9x^3 + 8 = 0$.

110. $9x^4 - 13x^2 + 4 = 0$.

111. $4x^4 - 17x^2 + 4 = 0$.

112. $2x^4 - 5x^2 + 2 = 0$.

113. $2x^{\frac{1}{2}} - 3x^{\frac{1}{4}} + 1 = 0$.

114. $3\sqrt[3]{x} - 5\sqrt[6]{x} + 2 = 0$.

115. $6\sqrt{x} - 3\sqrt[4]{x} - 45 = 0$.

116. $21\sqrt[3]{x^2} - 5\sqrt[3]{x} - 74 = 0$.

117. $3\sqrt{x^5} + 4\sqrt[4]{x^5} - 20 = 0$.

118. $2x^6 - 19x^3 + 24 = 0$.

119. $x^4 - 1 = 0$.

120. $x^6 - 1 = 0.$

121. $x^{\frac{2}{3}} + 8x^{\frac{1}{3}} - 9 = 0.$

122. $16x^{\frac{4}{5}} - 17x^{\frac{2}{5}} + 1 = 0.$

123. $x^{-\frac{1}{2}} + 5x^{-\frac{1}{4}} - 14 = 0.$

124. $4x^{-\frac{2}{3}} - 3x^{-\frac{1}{3}} - 27 = 0.$

125. $x^{2n} + 3x^n - 4 = 0.$

126. $3x^{\frac{1}{3}} - 2ax^{\frac{1}{6}} - a^2 = 0.$

127. $\sqrt{2x} - \sqrt[4]{2x} - 2 = 0.$

Solve:

128. $\sqrt{x+4} + \sqrt{3x+1} = \sqrt{9x+4}.$

129. $\sqrt{5x+1} + 2\sqrt{4x-3} = 10\sqrt{x-2}.$

130. $2\sqrt{x+2} - 3\sqrt{3x-5} + \sqrt{5x+1} = 0.$

131. $\sqrt{11-x} + \sqrt{8-2x} - \sqrt{21+2x} = 0.$

Expand:

132. $(a^{\frac{1}{2}} - x^{\frac{2}{3}})^9;\ (3a^{\frac{1}{3}} - 2b^2)^5;\ (a^{\frac{2}{3}} - \frac{1}{2}b^{\frac{1}{2}})^7;$
$(2 - \frac{1}{3}x^2)^6;\ (3 - \frac{1}{2}x^{\frac{1}{2}})^7;\ (x^2 - x^{-2})^8.$

133. $(x^2 - \frac{1}{4}y^3)^5;\ (2x^2 + \sqrt{3x})^4;\ (\sqrt{a^2+1} + a^3)^5;$
$(1 + 2x - x^2 - x^3)^3.$

134. Expand to four terms

$(1-3x)^{-\frac{1}{3}};\ (1-4x^2)^{-\frac{2}{3}};\ (1-\frac{3}{2}x^{\frac{1}{2}})^{\frac{2}{3}};\ (a-2x^{-\frac{1}{3}})^{-5}.$

135. Find the eighty-seventh term of $(2x-y)^{90}$.

136. Resolve into partial fractions

$$\frac{3-2x}{1-3x+2x^2};\quad \frac{3-2x}{(1-x)(1-3x)};\quad \frac{1}{1-x^3}.$$

137. Expand to five terms $\dfrac{3-2x}{1-3x+2x^2}.$

CHAPTER XXVI.

LOGARITHMS.

411. If numbers are regarded as **powers of ten,** the **exponents** of the powers are the **Common or Briggs Logarithms** of the numbers.

If A and B denote two numbers, a and b their logarithms, then $10^a = A$, $10^b = B$; or, written in logarithmic form, $\log A = a$, $\log B = b$.

412. The logarithm of a product is found by adding the logarithms of its factors.

For $$A \times B = 10^a \times 10^b = 10^{a+b}. \qquad (\S\ 244)$$

Therefore, $$\log (A \times B) = a + b = \log A + \log B.$$

413. The logarithm of a quotient is found by subtracting the logarithm of the divisor from that of the dividend.

For $$\frac{A}{B} = \frac{10^a}{10^b} = 10^{a-b}. \qquad (\S\ 250)$$

Therefore, $$\log \frac{A}{B} = a - b = \log A - \log B.$$

414. The logarithm of a power of a number is found by multiplying the logarithm of the number by the exponent of the power.

For $$A^n = (10^a)^n = 10^{na}. \qquad (\S\ 251)$$

Therefore, $$\log A^n = na = n \log A.$$

415. The logarithm of the root of a number is found by dividing the logarithm of the number by the index of the root.

For $$\sqrt[n]{A} = \sqrt[n]{10^a} = 10^{\frac{a}{n}}. \qquad (\S\ 251)$$

Therefore, $$\log \sqrt[n]{A} = \frac{a}{n} = \frac{\log A}{n}.$$

416. The logarithms of 1, 10, 100, etc., and of 0.1, 0.01, 0.001, etc., are integral numbers. The logarithms of all other numbers are fractions.

Since
$$\begin{array}{ll} 10^0 = 1, & 10^{-1} \left(= \tfrac{1}{10}\right) = 0.1, \\ 10^1 = 10, & 10^{-2} \left(= \tfrac{1}{100}\right) = 0.01, \\ 10^2 = 100, & 10^{-3} \left(= \tfrac{1}{1000}\right) = 0.001, \end{array}$$

therefore,
$$\begin{array}{ll} \log 1 = 0, & \log 0.1 = -1, \\ \log 10 = 1, & \log 0.01 = -2, \\ \log 100 = 2, & \log 0.001 = -3. \end{array}$$

Hence, the common logarithms of all numbers between

1 and 10 will be $0 +$ a fraction,
10 and 100 will be $1 +$ a fraction,
100 and 1000 will be $2 +$ a fraction,
1 and 0.1 will be $-1 +$ a fraction,
0.1 and 0.01 will be $-2 +$ a fraction,
0.01 and 0.001 will be $-3 +$ a fraction.

417. If the number is less than 1, the logarithm is negative (§ 416), but is written in such a form that the *fractional part* is always *positive.*

418. Every logarithm, therefore, consists of two parts: a positive or negative integral number, which is called the **characteristic**, and a *positive* decimal fraction, which is called the **mantissa.**

Thus, in the logarithm 3.52184, the integral number 3 is the characteristic, and the fraction .52184, the mantissa In the logarithm 0.78256 − 2, which is sometimes written $\bar{2}.78256$, the integral number − 2 is the characteristic, and the fraction .78256 is the mantissa.

419. If the logarithm has a negative characteristic, it is customary to change its form by adding 10, or a multiple of 10, to the characteristic, and then indicating the subtraction of the same number from the result.

Thus, the logarithm $\bar{2}.78256$ is changed to 8.78256 − 10 by adding 10 to the characteristic and writing − 10 after the result. The logarithm $\overline{13}.92732$ is changed to 7.92732 − 20 by adding 20 to the characteristic and writing − 20 after the result.

420. The following rules are derived from § 416.

Rule 1. If the number is *greater than* 1, make the *characteristic* of the logarithm *one unit less* than the number of figures on the left of the decimal point.

Rule 2. If the number is *less than* 1, make the *characteristic* of the logarithm *negative*, and *one unit more* than the number of zeros between the decimal point and the first significant figure of the given number.

Rule 3. If the *characteristic* of a given logarithm is *positive*, make the number of figures in the integral part of the corresponding number *one more* than the number of units in the characteristic.

Rule 4. If the *characteristic* is *negative*, make the number of zeros between the decimal point and the first significant figure of the corresponding number *one less* than the number of units in the characteristic.

Thus, the characteristic of log 7849.27 is 3; the characteristic of log 0.037 is -2, or $8.00000-10$. If the characteristic is 4, the corresponding number has five figures in its integral part. If the characteristic is -3, that is, $7.00000-10$, the corresponding fraction has two zeros between the decimal point and the first significant figure.

421. The *mantissa* of the common logarithm of any integral number, or decimal fraction, depends only upon the digits of the number, and is unchanged so long as the *sequence of the digits* remains the same.

For, changing the position of the decimal point in a number is equivalent to multiplying or dividing the number by a power of 10. Its common logarithm, therefore, will be increased or diminished by the *exponent* of that power of 10; and since this exponent is *integral*, the *mantissa*, or decimal part of the logarithm, will be unaffected.

$$\begin{array}{rlrl}
\text{Thus, } 271{,}960 &= 10^{5.48451}, & 2.7196 &= 10^{0.48451},\\
27{,}196 &= 10^{4.43451}, & 0.27196 &= 10^{9.48451-10},\\
2719.6 &= 10^{3.48451}, & 0.027196 &= 10^{8.48451-10},\\
271.96 &= 10^{2.48451}, & 0.0027196 &= 10^{7.48451-10},\\
27.196 &= 10^{1.48451}, & 0.00027196 &= 10^{6.48461-10}.
\end{array}$$

One advantage of using the number *ten* as the *base* of a system of logarithms consists in the fact that the *mantissa* depends only on the *sequence of digits*, and the *characteristic* on the *position of the decimal point.*

422. The *characteristic* of the common logarithm of a number can be found by the rules given in § 420; but the *mantissa* of the logarithm is found by means of a *Table of Logarithms.*

423. A Five-place Table of Logarithms. In this table (pp. 389–407) the vertical columns headed **N** contain numbers, the columns headed **0**, **1**, **2**, **3**, etc., contain the logarithms, and the columns headed **D** contain the tabular differences of the logarithms. On page 389 both the characteristic and the mantissa are printed. On pages 390–407 the mantissa only is printed, and the *first two figures* of the mantissa are printed in the *left-hand column only*.

A star prefixed to the last three figures of a logarithm indicates that the *first two figures are in the line below.*

The fractional part of a logarithm can be expressed only approximately, and in a five-place table all figures that follow the fifth are rejected. *Whenever the sixth figure is 5 or more, the fifth figure is increased by 1.*

Thus, if the mantissa of a logarithm written to seven places is 5328732, it is written in a five-place table 53287. If the mantissa is 5328751, it is written in a five-place table 53288.

To Find the Logarithm of a Given Number.

424. If the given number consists of *one or two* figures, the logarithm is given on page 389. If zeros follow the significant figures, or if the number is a proper decimal fraction, the characteristic is determined by § 420.

425. If the given number has *three* significant figures, the number will be found in the column headed **N** (pp. 390–407), and the mantissa of its logarithm in the next column to the right, and on the same line. Thus,

Page 390. $\log 145 = 2.16137$, $\log 14{,}500 = 4.16137$.
Page 395. $\log 364 = 2.56110$, $\log 0.0364 = 8.56110 - 10$.
Page 402. $\log 716 = 2.85491$, $\log 0.716 = 9.85491 - 10$.
Page 406. $\log 926 = 2.96661$, $\log 9260 = 3.96661$.

426. If the given number has *four* significant figures, the first three figures will be found in the column headed **N**, and the fourth figure at the top of the page in the line containing the figures **0, 1, 2, 3,** etc. The mantissa will be found in the column headed by the fourth figure, and on the same line with the first three figures. Thus,

Page 403. log 7682 = 3.88547, log 76.85 = 1.88564.
Page 406. log 93,280 = 4.96979, log 0.9468 = 9.97626 − 10.

427. If the given number has *five or more* significant figures, a process called **interpolation** is required.

Interpolation is based on the *assumption* that between two consecutive mantissas of the table the change in the mantissa is directly proportional to the change in the number.

1. Find the logarithm of 34,237.

The required mantissa is (§ 421) the same as the mantissa for 3423.7; therefore, it will be found by adding to the mantissa for 3423 seven tenths of the difference between the mantissas for 3423 and 3424.

The mantissa for 3423 is 53441.
The difference between the mantissas for 3423 and 3424 is 12.
Hence, the mantissa for 3423.7 is 53441 + (0.7 of 12) = 53449.
Therefore, the logarithm of 34,237 is 4.53449.

2. Find the logarithm of 0.0015764.

The required mantissa is the same as the mantissa for 1576.4.
The mantissa for 1576 is 19756.
The difference between the mantissas for 1576 and 1577 is 27.
Hence, the mantissa for 1576.4 is 19756 + (0.4 of 27) = 19767.
Therefore, the logarithm of 0.0015764 is 7.19767 − 10.

3. Find the logarithm of 32.6708.

The required mantissa is the same as the mantissa for 3267.08.
The mantissa for 3267 is 51415.
The difference between the mantissas for 3267 and 3268 is 13.
Hence, the mantissa for 3267.08 is 51415 + (0.08 of 13) = 51416.
Therefore, the logarithm of 32.6708 is 1.51416.

428. When the fraction of a unit in the part to be added to the mantissa for four figures is less than 0.5, it is to be neglected; when it is 0.5 or more than 0.5, it is to be taken as one unit.

Thus, in example 1, § 427, the part to be added to the mantissa for 3423 is 8.4, and the .4 is rejected. In example 2, the part to be added to the mantissa for 1576 is 10.8, and 11 is added.

To Find the Antilogarithm; that is, the Number Corresponding to a Given Logarithm.

429. If the given mantissa can be found in the table, the first three figures of the required number will be found in the same line with the mantissa in the column headed **N**, and the fourth figure at the top of the column containing the mantissa.

The position of the decimal point is determined by the characteristic (§ 420).

1. Find the number corresponding to the logarithm 0.92002.

Page 404. The number for the mantissa 92002 is 8318.
The characteristic is 0; hence (§ 420), the number is 8.318.

2. Find the number corresponding to the logarithm 6.09167.

Page 390. The number for the mantissa 09167 is 1235.
The characteristic is 6; hence (§ 420), the number is 1,235,000.

3. Find the number corresponding to the logarithm 2.51055.

Page 394. The number for the mantissa 51055 is 3240.
The characteristic is 2; hence (§ 420), the number is 324.

4. Find the number corresponding to the logarithm 7.50325 – 10.

Page 394. The number for the mantissa 50325 is 3186.
The characteristic is – 3; hence (§ 420), the number is 0.003186.

430. If the given mantissa cannot be found in the table, find in the table the two adjacent mantissas between which the given mantissa lies, and the four figures corresponding to the smaller of these two mantissas will be the first four significant figures of the required number. If more than four figures are desired, they may be found by interpolation, as in the following examples:

1. Find the number corresponding to the logarithm 1.48762.

Here the two adjacent mantissas of the table, between which the given mantissa 48762 lies, are found to be (p. 394) 48756 and 48770. The corresponding numbers are 3073 and 3074. The smaller of these, 3073, contains the first four significant figures of the required number.

The difference between the two adjacent mantissas is 14, and the difference between the corresponding numbers is 1.

The difference between the smaller of the two adjacent mantissas, 48756, and the given mantissa, 48762, is 6. Therefore, the number to be annexed to 3073 is $\frac{6}{14}$ of $1 = 0.428$, and the fifth significant figure of the required number is 4.

Hence, the required number is 30.734.

2. Find the number corresponding to the logarithm $7.82326 - 10$.

Here the two adjacent mantissas of the table, between which the given mantissa 82326 lies, are found to be (p. 401) 82321 and 82328. The corresponding numbers are 6656 and 6657. The smaller of these, 6656, contains the first four significant figures of the required number.

The difference between the two adjacent mantissas is 7, and the difference between the corresponding numbers is 1.

The difference between the smaller mantissa, 82321, and the given mantissa, 82326, is 5. Therefore, the number to be annexed to 6656 is $\frac{5}{7}$ of $1 = 0.7$, and the fifth significant figure of the required number is 7.

Hence, the required number is 0.0066567.

Cologarithms.

431. The logarithm of the reciprocal of a number is called the **cologarithm** of the number.

If A denotes any number, then

$$\text{colog } A = \log \frac{1}{A} = \log 1 - \log A \ (\S\ 413) = -\log A \ (\S\ 416).$$

Hence, the cologarithm of a number is equal to the logarithm of the number with the minus sign prefixed, which sign affects *the entire logarithm.*

In order to avoid a negative mantissa in the cologarithm, it is customary to substitute for $-\log A$ its equivalent $(10 - \log A) - 10$.

Hence, the cologarithm of a number is found by subtracting the logarithm of the number from 10, and then annexing -10 to the remainder.

The best way to perform the subtraction is to begin on the left and subtract from 9 each significant figure of $\log A$ except the last, and subtract this from 10.

If $\log A$ is greater in absolute value than 10 and less than 20, then in order to avoid a negative mantissa it is necessary to write $-\log A$ in the form

$$(20 - \log A) - 20.$$

So that, in this case, colog A is found by subtracting $\log A$ from 20, and then annexing -20 to the remainder.

1. Find the cologarithm of 4007.

Page 396.

$$\begin{array}{rl} & 10. \qquad\quad -10 \\ \log 4007 = & \underline{3.60282 \qquad\qquad} \\ \text{colog } 4007 = & 6.39718 - 10 \end{array}$$

2. Find the cologarithm of 103,992,000,000.

Page 390.

$$\begin{array}{rl} & 20. \qquad\quad -20 \\ \log 103992000000 = & \underline{11.01700 \qquad\qquad} \\ \text{colog } 103992000000 = & 8.98300 - 20 \end{array}$$

If the characteristic of log A is negative, then the subtrahend, -10 or -20, will vanish in finding the value of colog A.

3. Find the cologarithm of 0.004007.

Page 396.

$$\begin{array}{rl} & 10. \qquad\quad -10 \\ \log 0.004007 = & 7.60282 - 10 \\ \cline{2-2} \text{colog } 0.004007 = & 2.39718 \end{array}$$

With practice, the cologarithm of a number can be taken from the table as rapidly as the logarithm itself.

By using cologarithms the inconvenience of subtracting the logarithm of a divisor is avoided. For dividing by a number is equivalent to multiplying by its reciprocal. Hence, instead of subtracting the logarithm of a divisor its cologarithm may be added.

Exercise 136.

Find the logarithm of:

1. 6170.
2. 0.617.
3. 2867.
4. 85.76.
5. 296.8.
6. 7004.
7. 0.8694.
8. 0.5908.
9. 73,243.
10. 67.3208.
11. 18.5283.
12. 0.0042003.

Find the cologarithm of:

13. 72,433.
14. 802.376.
15. 15.7643.
16. 869.278.
17. 154,000.
18. 70.0426.
19. 0.002403.
20. 0.000777.
21. 0.051828.

Find the antilogarithm of:

22. 2.47246.
23. 7.89081.
24. 2.91221.
25. 1.26784.
26. 3.79029.
27. 5.18752.
28. 9.79029 − 10.
29. 7.62328 − 10.
30. 6.15465 − 10.

Computation by Logarithms.

1. Find the value of x, if $x = 72{,}214 \times 0.08203$.

Page 402. $\log 72214 = 4.85862$
Page 404. $\log 0.08203 = \underline{8.91397 - 10}$
By § 412. $\log x = 3.77259$
Page 399. $x = 5923.6$

In using a five-place table the numbers corresponding to mantissas may be carried to five significant figures, and in the first part of the table to six figures.

2. Find the product of $-908.4 \times 0.05392 \times 2.117$.

Page 406. $\log 908.4 = 2.95828^n$
Page 398. $\log 0.05392 = 8.73175 - 10$
Page 392. $\log 2.117 = \underline{0.32572}$
Page 390. $2.01575 = \log 103.69$. -103.69. *Ans.*

When any of the factors are *negative*, find their logarithms, without regard to the signs; write n after the logarithm that corresponds to a negative number. If the number of logarithms so marked is *odd*, the product is *negative*; if *even*, the product is *positive*.

3. Find the value of x, if $x = 5250 \div 23{,}487$.

Page 398. $\log 5250 = 3.72016$
Page 392. $\text{colog } 23487 = \underline{5.62917 - 10}$
Page 392. $\log x = 9.34933 - 10 = \log 0.22353.$
$\therefore x = 0.22353$

4. Find the value of x, if $x = \dfrac{7.56 \times 4667 \times 567}{899.1 \times 0.00337 \times 23435}$.

Page 403. $\log 7.56 = 0.87852$
Page 397. $\log 4667 = 3.66904$
Page 399. $\log 567 = 2.75358$
Page 405. $\text{colog } 899.1 = 7.04619 - 10$
Page 394. $\text{colog } 0.00337 = 2.47237$
Page 392. $\text{colog } 23435 = \underline{5.63013 - 10}$
Page 393. $\log x = 2.44983 = \log 281.73.$
$\therefore x = 281.73.$

5. Find the cube of 376.

$$\begin{array}{lll}
\text{Page 395.} & \log 376 & = 2.57519 \\
\text{Multiply by 3 (§ 414),} & & \,\underline{\qquad\quad 3} \\
\text{Page 398.} & \log 376^3 & = 7.72557 = \log 53159000. \\
 & \therefore 376^3 & = 53{,}159{,}000.
\end{array}$$

6. Find the square of 0.003278.

$$\begin{array}{lll}
\text{Page 394.} & \log 0.003278 & = 7.51561 - 10 \\
\text{Multiply by 2 (§ 414),} & & \,\underline{\qquad\qquad\quad 2} \\
\text{Page 390.} & \log 0.003278^2 & = 15.03122 - 20 = \log 0.000010745. \\
 & \therefore 0.003278^2 & = 0.000010745.
\end{array}$$

7. Find the square root of 8322.

$$\begin{array}{lll}
\text{Page 404.} & \log 8322 & = 3.92023 \\
\text{Divide by 2 (§ 415),} & & 2\,\underline{)\,3.92023} \\
\text{Page 406.} & \log \sqrt{8322} & = 1.96012 = \log 91.226. \\
 & \therefore \sqrt{8322} & = 91.226.
\end{array}$$

If the given number is a proper fraction, its logarithm will have as a subtrahend 10 or a multiple of 10. In this case, before dividing the logarithm by the index of the root, both the subtrahend and the number preceding the mantissa should be increased by such a number as will make the subtrahend, when divided by the index of the root, 10 or a multiple of 10.

8. Find the square root of 0.000043641.

$$\begin{array}{lll}
\text{Page 396.} & \log 0.000043641 & = 5.63989 - 10 \\
 & & \;\underline{10. \qquad\;\; - 10} \\
\text{Divide by 2 (§ 415),} & & 2\,)\,15.63989 - 20 \\
\text{Page 401.} & \log \sqrt{0.000043641} & = 7.81995 - 10 = \log 0.0066062. \\
 & \therefore \sqrt{0.000043641} & = 0.0066062.
\end{array}$$

9. Find the sixth root of 0.076553.

$$\begin{array}{lll}
\text{Page 403.} & \log 0.076553 & = 8.88397 - 10 \\
 & & \;\underline{50. \qquad\;\; - 50} \\
\text{Divide by 6 (§ 415),} & & 6\,)\,58.88397 - 60 \\
\text{Page 400.} & \log \sqrt[6]{0.076553} & = 9.81400 - 10 = \log 0.65163. \\
 & \therefore \sqrt[6]{0.076553} & = 0.65163.
\end{array}$$

10. Find the value of $\sqrt[5]{\dfrac{3.1416 \times 4771.21 \times 2.7183^{\frac{1}{2}}}{30.103^4 \times 0.4343^{\frac{1}{2}} \times 69.897^4}}$.

$$\begin{array}{rll}
\log 3.1416 & = 0.49715 & = 0.49715 \\
\log 4771.21 & = 3.67863 & = 3.67863 \\
\tfrac{1}{2} \log 2.7183 & = \tfrac{1}{2}(0.43430) & = 0.21715 \\
4 \operatorname{colog} 30.103 & = 4(8.52139 - 10) & = 4.08556 - 10 \\
\tfrac{1}{2} \operatorname{colog} 0.4343 & = \tfrac{1}{2}(0.36221) & = 0.18111 \\
4 \operatorname{colog} 69.897 & = 4(8.15554 - 10) & = \underline{2.62216 - 10} \\
 & & \ 11.28176 - 20 \\
 & & \ 30. \qquad\;\; - 30 \\
 & & 5\,\overline{)\,41.28176 - 50} \\
 & & \ 8.25635 - 10 \\
 & & = \log 0.018045.
\end{array}$$

11. Find the value of x in $81^x = 10$.

$$81^x = 10.$$

$$\therefore \log 81^x = \log 10.$$

$$x \log 81 = \log 10. \qquad (\S\ 414)$$

$$x = \frac{\log 10}{\log 81} = \frac{1.00000}{1.90849} = 0.52397.$$

Exercise 137.

Find the value of:

1. 849.7×0.7834.
2. 3.709×0.08673.
3. 83.75×0.009376.
4. 8593×0.0008974.
5. -0.007634×6457.
6. -0.07843×48.66.
7. -0.8734×0.4378.
8. $-7.384 \times (-5.837)$.
9. 4657×3145.
10. $6078 \div 8703$.
11. $8.326 \div 0.1978$.
12. $0.6539 \div 0.9761$.
13. $-2.567 \div 0.6785$.
14. $(39.47 \times 5.938) \div 76.54$.
15. $(5674 \times 0.763) \div 0.9803$.
16. $357 \div (7069 \times 0.07948)$.
17. $8.9 \div (17.81 \times 0.002831)$.
18. $51.98 \div (81.71 \times 0.0008002)$.

19. $\dfrac{79.32 \times 0.005763 \times 0.8064}{0.4273 \times 0.8462 \times 0.01}$.

20. $\dfrac{72.56 \times 0.0005723 \times 8979}{42.28 \times 4.745 \times 0.006158}$.

21. $\dfrac{0.01723 \times 34.29 \times 0.5477}{0.07692 \times 37.69 \times 0.7733}$.

22. $\dfrac{7.126 \times 0.7213 \times 0.7583}{0.4684 \times 7.385 \times 0.9673}$.

23. $\dfrac{2.057 \times 77.12 \times 0.004896 \times 4.771}{7.582 \times 97.33 \times 0.008697 \times 0.4963}$.

24. 5.03^3.

25. 15.01^5.

26. 76.85^4.

27. 8.713^2.

28. 0.6787^3.

29. 0.9679^5.

30. 0.07867^3.

31. 0.008546^2.

32. $9.068^{\frac{1}{4}}$.

33. $0.0635^{\frac{1}{3}}$.

34. $0.008721^{\frac{1}{5}}$.

35. $0.6543^{\frac{1}{2}}$.

36. $\sqrt[3]{\dfrac{83.25 \times 8375 \times 0.008576}{0.0327 \times 687.5 \times 0.005003}}$.

37. $\sqrt[5]{\dfrac{4.163^2 \times 17.74 \times 0.7183^{\frac{1}{3}}}{3.013^2 \times 34.34 \times 0.08137^{\frac{1}{2}}}}$.

38. $\sqrt{\dfrac{0.7132 \times 9.245 \times 0.5477^2}{76.93 \times 0.000173^{\frac{1}{3}} \times 0.01}}$.

39. $\sqrt[3]{\dfrac{65.02 \times 0.002753 \times 97.98}{7.298^2 \times 0.04754 \times 8.156^2}}$.

40. $\sqrt[3]{\dfrac{0.6012 \times \sqrt{0.6012} \times \sqrt[3]{0.6012}}{0.5926 \times \sqrt{0.5926} \times \sqrt[3]{0.5926}}}$.

Find x from the equation :

41. $5^x = 10$.

42. $4^x = 20$.

43. $7^x = 40$.

44. $(1.3)^x = 4.2$.

45. $(0.4)^{-x} = 3$.

46. $(0.9)^{-x} = 2$.

Compound Interest and Annuities.

432. The amount of $\$P$ at compound interest at r per cent

$$\begin{array}{ll} \text{for 1 year is} & P(1+r); \\ \text{for 2 years is} & P(1+r)^2; \\ \text{for } n \text{ years is} & P(1+r)^n. \end{array}$$

Hence, if the amount for n years is represented by A,

$$A = P(1+r)^n.$$

NOTE. If the interest is compounded semi-annually,

$$A = P(1 + \tfrac{1}{2}r)^{2n}.$$

Find the amount of $150 for 6 years at 4% compound interest.

$$A = P(1+r)^n = 150\,(1.04)^6.$$

$$\begin{array}{ll} \log 150 & = 2.17609 \\ \log 1.04^6 & = 0.10218 \\ \hline \log A & = 2.27827 = \log 189.79. \end{array}$$

Hence, the required amount is $189.79.

433. The present worth, P, of $\$A$, payable in n years at r per cent, must just amount in n years to A.

Hence,
$$P = \frac{A}{(1+r)^n}. \qquad (\S\ 432)$$

434. An **annuity** is a sum of money to be paid at regular intervals of time, as years, half years, quarter years.

435. To find the present value of an annuity of $\$A$ per annum for n years, at r per cent.

The present value of the 1st payment is $\dfrac{A}{(1+r)}$; (§ 433)

The present value of the 2d payment is $\frac{A}{(1+r)^2}$;

The present value of the nth payment is $\frac{A}{(1+r)^n}$.

Hence, the present value of all the payments is

$$\frac{A}{(1+r)}+\frac{A}{(1+r)^2}+\cdots\cdots+\frac{A}{(1+r)^n}$$

$$=\frac{\frac{A}{1+r}\left[1-\left(\frac{1}{1+r}\right)^n\right]}{1-\frac{1}{1+r}}. \qquad (\S\ 365)$$

Multiply both numerator and denominator by $1+r$,

$$=\frac{A\left[1-\frac{1}{(1+r)^n}\right]}{r}.$$

$$=\frac{A}{r}\left[\frac{(1+r)^n-1}{(1+r)^n}\right].$$

Find the present value of an annuity of \$500 for 5 years, if money is worth 4%.

$$P=\frac{A}{r}\left[\frac{(1+r)^n-1}{(1+r)^n}\right]=\frac{500}{0.04}\left(\frac{1.04^5-1}{1.04^5}\right).$$

$$\log 1.04^5=5\times 0.01703=0.08515=\log 1.2166.$$

$$\therefore P=\frac{500}{0.04}\times\frac{0.2166}{1.2166}.$$

$$\begin{aligned}
\log 500 &= 2.69897\\
\log 0.2166 &= 9.33566-10\\
\operatorname{colog} 0.04 &= 1.39794\\
\operatorname{colog} 1.2166 &= \underline{9.91485-10}\\
&\ 3.34742 \qquad = \log 2225.5.
\end{aligned}$$

Therefore, the present value of the annuity is \$2225.50.

EXERCISE 138.

Find the amount at compound interest:

1. Of $8764 for 9 years at 5%.
2. Of $16,470 for 17 years at 3½%.
3. Of $12,000 for 12 years at 4%.
4. Of $976.45 for 9 years 6 months at 4½%.

Find the principal that will:

5. Amount to $1200 in 7 years at 5% compound interest.
6. Amount to $18,740 in 12 years at 4% compound interest.
7. Amount to $847.55 in 5 years 3 months at 3½% compound interest.

Find the rate of compound interest:

8. If $1296 amounts to $1576.75 in 5 years.
9. If $4830 amounts to $6472.70 in 6 years.
10. If $4625 amounts to $7404.80 in 12 years.
11. In what time at 3½% will $2225 amount to $3225 at compound interest?
12. In what time at 5% will $1640 amount to $3000, interest being compounded semi-annually?

Find the present value of an annuity:

13. Of $750 for 12 years, if money is worth 4%.
14. Of $1200 for 10 years, if money is worth 5½%.
15. Of $1875 for 6 years, if money is worth 4%.
16. Of $3200 for 14 years, if money is worth 3½%.
17. Of $2500 for 8 years, if money is worth 3%.
18. Of $612.50 for 18 years, if money is worth 3½%.

A TABLE

OF THE

COMMON LOGARITHMS

OF

NUMBERS

LOGARITHMS OF NUMBERS TO 100.

1	0.00000	26	1.41497	51	1.70757	76	1.88081
2	0.30103	27	1.43136	52	1.71600	77	1.88649
3	0.47712	28	1.44716	53	1.72428	78	1.89209
4	0.60206	29	1.46240	54	1.73239	79	1.89763
5	0.69897	30	1.47712	55	1.74036	80	1.90309
6	0.77815	31	1.49136	56	1.74819	81	1.90849
7	0.84510	32	1.50515	57	1.75587	82	1.91381
8	0.90309	33	1.51851	58	1.76343	83	1.91908
9	0.95424	34	1.53148	59	1.77085	84	1.92428
10	1.00000	35	1.54407	60	1.77815	85	1.92942
11	1.04139	36	1.55630	61	1.78533	86	1.93450
12	1.07918	37	1.56820	62	1.79239	87	1.93952
13	1.11394	38	1.57978	63	1.79934	88	1.94448
14	1.14613	39	1.59106	64	1.80618	89	1.94939
15	1.17609	40	1.60206	65	1.81291	90	1.95424
16	1.20412	41	1.61278	66	1.81954	91	1.95904
17	1.23045	42	1.62325	67	1.82607	92	1.96379
18	1.25527	43	1.63347	68	1.83251	93	1.96848
19	1.27875	44	1.64345	69	1.83885	94	1.97313
20	1.30103	45	1.65321	70	1.84510	95	1.97772
21	1.32222	46	1.66276	71	1.85126	96	1.98227
22	1.34242	47	1.67210	72	1.85733	97	1.98677
23	1.36173	48	1.68124	73	1.86332	98	1.99123
24	1.38021	49	1.69020	74	1.86923	99	1.99564
25	1.39794	50	1.69897	75	1.87506	100	2.00000

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100	00000	043	087	130	173	217	260	303	346	389	43
101	432	475	518	561	604	647	689	732	775	817	43
102	860	903	945	988	*030	*072	*115	*157	*199	*242	42
103	01284	326	368	410	452	494	536	578	620	662	42
104	703	745	787	828	870	912	953	995	*036	*078	42
105	02119	160	202	243	284	325	366	407	449	490	41
106	531	572	612	653	694	735	776	816	857	898	41
107	938	979	*019	*060	*100	*141	*181	*222	*262	*302	40
108	03342	383	423	463	503	543	583	623	663	703	40
109	743	782	822	862	902	941	981	*021	*060	*100	40
110	04139	179	218	258	297	336	376	415	454	493	39
111	532	571	610	650	689	727	766	805	844	883	39
112	922	961	999	*038	*077	*115	*154	*192	*231	*269	39
113	05308	346	385	423	461	500	538	576	614	652	38
114	690	729	767	805	843	881	918	956	994	*032	38
115	06070	108	145	183	221	258	296	333	371	408	38
116	446	483	521	558	595	633	670	707	744	781	37
117	819	856	893	930	967	*004	*041	*078	*115	*151	37
118	07188	225	262	298	335	372	408	445	482	518	37
119	555	591	628	664	700	737	773	809	846	882	36
120	918	954	990	*027	*063	*099	*135	*171	*207	*243	36
121	08279	314	350	386	422	458	493	529	565	600	36
122	636	672	707	743	778	814	849	884	920	955	35
123	991	*026	*061	*096	*132	*167	*202	*237	*272	*307	35
124	09342	377	412	447	482	517	552	587	621	656	35
125	691	726	760	795	830	864	899	934	968	*003	35
126	10037	072	106	140	175	209	243	278	312	346	34
127	380	415	449	483	517	551	585	619	653	687	34
128	721	755	789	823	857	890	924	958	992	*025	34
129	11059	093	126	160	193	227	261	294	327	361	34
130	394	428	461	494	528	561	594	628	661	694	33
131	727	760	793	826	860	893	926	959	992	*024	33
132	12057	090	123	156	189	222	254	287	320	352	33
133	385	418	450	483	516	548	581	613	646	678	33
134	710	743	775	808	840	872	905	937	969	*001	32
135	13033	066	098	130	162	194	226	258	290	322	32
136	354	386	418	450	481	513	545	577	609	640	32
137	672	704	735	767	799	830	862	893	925	956	32
138	988	*019	*051	*082	*114	*145	*176	*208	*239	*270	31
139	14301	333	364	395	426	457	489	520	551	582	31
140	613	644	675	706	737	768	799	829	860	891	31
141	922	953	983	*014	*045	*076	*106	*137	*168	*198	31
142	15229	259	290	320	351	381	412	442	473	503	31
143	534	564	594	625	655	685	715	746	776	806	30
144	836	866	897	927	957	987	*017	*047	*077	*107	30
145	16137	167	197	227	256	286	316	346	376	406	30
146	435	465	495	524	554	584	613	643	673	702	30
147	732	761	791	820	850	879	909	938	967	997	29
148	17026	056	085	114	143	173	202	231	260	289	29
149	319	348	377	406	435	464	493	522	551	580	29
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150	17609	638	667	696	725	754	782	811	840	869	29
151	898	926	955	984	*013	*041	*070	*099	*127	*156	29
152	18184	213	241	270	298	327	355	384	412	441	29
153	469	498	526	554	583	611	639	667	696	724	28
154	752	780	808	837	865	893	921	949	977	*005	28
155	19033	061	089	117	145	173	201	229	257	285	28
156	312	340	368	396	424	451	479	507	535	562	28
157	590	618	645	673	700	728	756	783	811	838	28
158	866	893	921	948	976	*003	*030	*058	*085	*112	27
159	20140	167	194	222	249	276	303	330	358	385	27
160	412	439	466	493	520	548	575	602	629	656	27
161	683	710	737	763	790	817	844	871	898	925	27
162	952	978	*005	*032	*059	*085	*112	*139	*165	*192	27
163	21219	245	272	299	325	352	378	405	431	458	27
164	484	511	537	564	590	617	643	669	696	722	26
165	748	775	801	827	854	880	906	932	958	985	26
166	22011	037	063	089	115	141	167	194	220	246	26
167	272	298	324	350	376	401	427	453	479	505	26
168	531	557	583	608	634	660	686	712	737	763	26
169	789	814	840	866	891	917	943	968	994	*019	26
170	23045	070	096	121	147	172	198	223	249	274	25
171	300	325	350	376	401	426	452	477	502	528	25
172	553	578	603	629	654	679	704	729	754	779	25
173	805	830	855	880	905	930	955	980	*005	*030	25
174	24055	080	105	130	155	180	204	229	254	279	25
175	304	329	353	378	403	428	452	477	502	527	25
176	551	576	601	625	650	674	699	724	748	773	25
177	797	822	846	871	895	920	944	969	993	*018	25
178	25042	066	091	115	139	164	188	212	237	261	24
179	285	310	334	358	382	406	431	455	479	503	24
180	527	551	575	600	624	648	672	696	720	744	24
181	768	792	816	840	864	888	912	935	959	983	24
182	26007	031	055	079	102	126	150	174	198	221	24
183	245	269	293	316	340	364	387	411	435	458	24
184	482	505	529	553	576	600	623	647	670	694	24
185	717	741	764	788	811	834	858	881	905	928	23
186	951	975	998	*021	*045	*068	*091	*114	*138	*161	23
187	27184	207	231	254	277	300	323	346	370	393	23
188	416	439	462	485	508	531	554	577	600	623	23
189	646	669	692	715	738	761	784	807	830	852	23
190	875	898	921	944	967	989	*012	*035	*058	*081	23
191	28103	126	149	171	194	217	240	262	285	307	23
192	330	353	375	398	421	443	466	488	511	533	23
193	556	578	601	623	646	668	691	713	735	758	22
194	780	803	825	847	870	892	914	937	959	981	22
195	29003	026	048	070	092	115	137	159	181	203	22
196	226	248	270	292	314	336	358	380	403	425	22
197	447	469	491	513	535	557	579	601	623	645	22
198	667	688	710	732	754	776	798	820	842	863	22
199	885	907	929	951	973	994	*016	*038	*060	*081	22
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200	30103	125	146	168	190	211	233	255	276	298	22
201	320	341	363	384	406	428	449	471	492	514	22
202	535	557	578	600	621	643	664	685	707	728	21
203	750	771	792	814	835	856	878	899	920	942	21
204	963	984	*006	*027	*048	*069	*091	*112	*133	*154	21
205	31175	197	218	239	260	281	302	323	345	366	21
206	387	408	429	450	471	492	513	534	555	576	21
207	597	618	639	660	681	702	723	744	765	785	21
208	806	827	848	869	890	911	931	952	973	994	21
209	32015	035	056	077	098	118	139	160	181	201	21
210	222	243	263	284	305	325	346	366	387	408	21
211	428	449	469	490	510	531	552	572	593	613	20
212	634	654	675	695	715	736	756	777	797	818	20
213	838	858	879	899	919	940	960	980	*001	*021	20
214	33041	062	082	102	122	143	163	183	203	224	20
215	244	264	284	304	325	345	365	385	405	425	20
216	445	465	486	506	526	546	566	586	606	626	20
217	646	666	686	706	726	746	766	786	806	826	20
218	846	866	885	905	925	945	965	985	*005	*025	20
219	34044	064	084	104	124	143	163	183	203	223	20
220	242	262	282	301	321	341	361	380	400	420	20
221	439	459	479	498	518	537	557	577	596	616	20
222	635	655	674	694	713	733	753	772	792	811	19
223	830	850	869	889	908	928	947	967	986	*005	19
224	35025	044	064	083	102	122	141	160	180	199	19
225	218	238	257	276	295	315	334	353	372	392	19
226	411	430	449	468	488	507	526	545	564	583	19
227	603	622	641	660	679	698	717	736	755	774	19
228	793	813	832	851	870	889	908	927	946	965	19
229	984	*003	*021	*040	*059	*078	*097	*116	*135	*154	19
230	36173	192	211	229	248	267	286	305	324	342	19
231	361	380	399	418	436	455	474	493	511	530	19
232	549	568	586	605	624	642	661	680	698	717	19
233	736	754	773	791	810	829	847	866	884	903	19
234	922	940	959	977	996	*014	*033	*051	*070	*088	18
235	37107	125	144	162	181	199	218	236	254	273	18
236	291	310	328	346	365	383	401	420	438	457	18
237	475	493	511	530	548	566	585	603	621	639	18
238	658	676	694	712	731	749	767	785	803	822	18
239	840	858	876	894	912	931	949	967	985	*003	18
240	38021	039	057	075	093	112	130	148	166	184	18
241	202	220	238	256	274	292	310	328	346	364	18
242	382	399	417	435	453	471	489	507	525	543	18
243	561	578	596	614	632	650	668	686	703	721	18
244	739	757	775	792	810	828	846	863	881	899	18
245	917	934	952	970	987	*005	*023	*041	*058	*076	18
246	39094	111	129	146	164	182	199	217	235	252	18
247	270	287	305	322	340	358	375	393	410	428	18
248	445	463	480	498	515	533	550	568	585	602	18
249	620	637	655	672	690	707	724	742	759	777	17
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250	39794	811	829	846	863	881	898	915	933	950	17
251	967	985	*002	*019	*037	*054	*071	*088	*106	*123	17
252	40140	157	175	192	209	226	243	261	278	295	17
253	312	329	346	364	381	398	415	432	449	466	17
254	483	500	518	535	552	569	586	603	620	637	17
255	654	671	688	705	722	739	756	773	790	807	17
256	824	841	858	875	892	909	926	943	960	976	17
257	993	*010	*027	*044	*061	*078	*095	*111	*128	*145	17
258	41162	179	196	212	229	246	263	280	296	313	17
259	330	347	363	380	397	414	430	447	464	481	17
260	497	514	531	547	564	581	597	614	631	647	17
261	664	681	697	714	731	747	764	780	797	814	17
262	830	847	863	880	896	913	929	946	963	979	16
263	996	*012	*029	*045	*062	*078	*095	*111	*127	*144	16
264	42160	177	193	210	226	243	259	275	292	308	16
265	325	341	357	374	390	406	423	439	455	472	16
266	488	504	521	537	553	570	586	602	619	635	16
267	651	667	684	700	716	732	749	765	781	797	16
268	813	830	846	862	878	894	911	927	943	959	16
269	975	991	*008	*024	*040	*056	*072	*088	*104	*120	16
270	43136	152	169	185	201	217	233	249	265	281	16
271	297	313	329	345	361	377	393	409	425	441	16
272	457	473	489	505	521	537	553	569	584	600	16
273	616	632	648	664	680	696	712	727	743	759	16
274	775	791	807	823	838	854	870	886	902	917	16
275	933	949	965	981	996	*012	*028	*044	*059	*075	16
276	44091	107	122	138	154	170	185	201	217	232	16
277	248	264	279	295	311	326	342	358	373	389	16
278	404	420	436	451	467	483	498	514	529	545	16
279	560	576	592	607	623	638	654	669	685	700	16
280	716	731	747	762	778	793	809	824	840	855	15
281	871	886	902	917	932	948	963	979	994	*010	15
282	45025	040	056	071	086	102	117	133	148	163	15
283	179	194	209	225	240	255	271	286	301	317	15
284	332	347	362	378	393	408	423	439	454	469	15
285	484	500	515	530	545	561	576	591	606	621	15
286	637	652	667	682	697	712	728	743	758	773	15
287	788	803	818	834	849	864	879	894	909	924	15
288	939	954	969	984	*000	*015	*030	*045	*060	*075	15
289	46090	105	120	135	150	165	180	195	210	225	15
290	240	255	270	285	300	315	330	345	359	374	15
291	389	404	419	434	449	464	479	494	509	523	15
292	538	553	568	583	598	613	627	642	657	672	15
293	687	702	716	731	746	761	776	790	805	820	15
294	835	850	864	879	894	909	923	938	953	967	15
295	982	997	*012	*026	*041	*056	*070	*085	*100	*114	15
296	47129	144	159	173	188	202	217	232	246	261	15
297	276	290	305	319	334	349	363	378	392	407	15
298	422	436	451	465	480	494	509	524	538	553	15
299	567	582	596	611	625	640	654	669	683	698	15
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300	47712	727	741	756	770	784	799	813	828	842	14
301	857	871	885	900	914	929	943	958	972	986	14
302	48001	015	029	044	058	073	087	101	116	130	14
303	144	159	173	187	202	216	230	244	259	273	14
304	287	302	316	330	344	359	373	387	401	416	14
305	430	444	458	473	487	501	515	530	544	558	14
306	572	586	601	615	629	643	657	671	686	700	14
307	714	728	742	756	770	785	799	813	827	841	14
308	855	869	883	897	911	926	940	954	968	982	14
309	996	*010	*024	*038	*052	*066	*080	*094	*108	*122	14
310	49136	150	164	178	192	206	220	234	248	262	14
311	276	290	304	318	332	346	360	374	388	402	14
312	415	429	443	457	471	485	499	513	527	541	14
313	554	568	582	596	610	624	638	651	665	679	14
314	693	707	721	734	748	762	776	790	803	817	14
315	831	845	859	872	886	900	914	927	941	955	14
316	969	982	996	*010	*024	*037	*051	*065	*079	*092	14
317	50106	120	133	147	161	174	188	202	215	229	14
318	243	256	270	284	297	311	325	338	352	365	14
319	379	393	406	420	433	447	461	474	488	501	14
320	515	529	542	556	569	583	596	610	623	637	14
321	651	664	678	691	705	718	732	745	759	772	14
322	786	799	813	826	840	853	866	880	893	907	13
323	920	934	947	961	974	987	*001	*014	*028	*041	13
324	51055	068	081	095	108	121	135	148	162	175	13
325	188	202	215	228	242	255	268	282	295	308	13
326	322	335	348	362	375	388	402	415	428	441	13
327	455	468	481	495	508	521	534	548	561	574	13
328	587	601	614	627	640	654	667	680	693	706	13
329	720	733	746	759	772	786	799	812	825	838	13
330	851	865	878	891	904	917	930	943	957	970	13
331	983	996	*009	*022	*035	*048	*061	*075	*088	*101	13
332	52114	127	140	153	166	179	192	205	218	231	13
333	244	257	270	284	297	310	323	336	349	362	13
334	375	388	401	414	427	440	453	466	479	492	13
335	504	517	530	543	556	569	582	595	608	621	13
336	634	647	660	673	686	699	711	724	737	750	13
337	763	776	789	802	815	827	840	853	866	879	13
338	892	905	917	930	943	956	969	982	994	*007	13
339	53020	033	046	058	071	084	097	110	122	135	13
340	148	161	173	186	199	212	224	237	250	263	13
341	275	288	301	314	326	339	352	364	377	390	13
342	403	415	428	441	453	466	479	491	504	517	13
343	529	542	555	567	580	593	605	618	631	643	13
344	656	668	681	694	706	719	732	744	757	769	13
345	782	794	807	820	832	845	857	870	882	895	13
346	908	920	933	945	958	970	983	995	*008	*020	13
347	54033	045	058	070	083	095	108	120	133	145	13
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353	777	790	802	814	827	839	851	864	876	888	12
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355	55023	035	047	060	072	084	096	108	121	133	12
356	145	157	169	182	194	206	218	230	242	255	12
357	267	279	291	303	315	328	340	352	364	376	12
358	388	400	413	425	437	449	461	473	485	497	12
359	509	522	534	546	558	570	582	594	606	618	12
360	630	642	654	666	678	691	703	715	727	739	12
361	751	763	775	787	799	811	823	835	847	859	12
362	871	883	895	907	919	931	943	955	967	979	12
363	991	*003	*015	*027	*038	*050	*062	*074	*086	*098	12
364	56110	122	134	146	158	170	182	194	205	217	12
365	229	241	253	265	277	289	301	312	324	336	12
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370	820	832	844	855	867	879	891	902	914	926	12
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373	171	183	194	206	217	229	241	252	264	276	12
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375	403	415	426	438	449	461	473	484	496	507	12
376	519	530	542	553	565	576	588	600	611	623	12
377	634	646	657	669	680	692	703	715	726	738	11
378	749	761	772	784	795	807	818	830	841	852	11
379	864	875	887	898	910	921	933	944	955	967	11
380	978	990	*001	*013	*024	*035	*047	*058	*070	*081	11
381	58092	104	115	127	138	149	161	172	184	195	11
382	206	218	229	240	252	263	274	286	297	309	11
383	320	331	343	354	365	377	388	399	410	422	11
384	433	444	456	467	478	490	501	512	524	535	11
385	546	557	569	580	591	602	614	625	636	647	11
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389	995	*006	*017	*028	*040	*051	*062	*073	*084	*095	11
390	59106	118	129	140	151	162	173	184	195	207	11
391	218	229	240	251	262	273	284	295	306	318	11
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393	439	450	461	472	483	494	506	517	528	539	11
394	550	561	572	583	594	605	616	627	638	649	11
395	660	671	682	693	704	715	726	737	748	759	11
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402	423	433	444	455	466	477	487	498	509	520	11
403	531	541	552	563	574	584	595	606	617	627	11
404	638	649	660	670	681	692	703	713	724	735	11
405	746	756	767	778	788	799	810	821	831	842	11
406	853	863	874	885	895	906	917	927	938	949	11
407	959	970	981	991	*002	*013	*023	*034	*045	*055	11
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413	595	606	616	627	637	648	658	669	679	690	11
414	700	711	721	731	742	752	763	773	784	794	10
415	805	815	826	836	847	857	868	878	888	899	10
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418	118	128	138	149	159	170	180	190	201	211	10
419	221	232	242	252	263	273	284	294	304	315	10
420	325	335	346	356	366	377	387	397	408	418	10
421	428	439	449	459	469	480	490	500	511	521	10
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423	634	644	655	665	675	685	696	706	716	726	10
424	737	747	757	767	778	788	798	808	818	829	10
425	839	849	859	870	880	890	900	910	921	931	10
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427	63043	053	063	073	083	094	104	114	124	134	10
428	144	155	165	175	185	195	205	215	225	236	10
429	246	256	266	276	286	296	306	317	327	337	10
430	347	357	367	377	387	397	407	417	428	438	10
431	448	458	468	478	488	498	508	518	528	538	10
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453	610	619	629	639	648	658	667	677	686	696	10
454	706	715	725	734	744	753	763	772	782	792	9
455	801	811	820	830	839	849	858	868	877	887	9
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457	992	*001	*011	*020	*030	*039	*049	*058	*068	*077	9
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459	181	191	200	210	219	229	238	247	257	266	9
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461	370	380	389	398	408	417	427	436	445	455	9
462	464	474	483	492	502	511	521	530	539	549	9
463	558	567	577	586	596	605	614	624	633	642	9
464	652	661	671	680	689	699	708	717	727	736	9
465	745	755	764	773	783	792	801	811	820	829	9
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471	302	311	321	330	339	348	357	367	376	385	9
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476	761	770	779	788	797	806	815	825	834	843	9
477	852	861	870	879	888	897	906	916	925	934	9
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481	215	224	233	242	251	260	269	278	287	296	9
482	305	314	323	332	341	350	359	368	377	386	9
483	395	404	413	422	431	440	449	458	467	476	9
484	485	494	502	511	520	529	538	547	556	565	9
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487	753	762	771	780	789	797	806	815	824	833	9
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491	108	117	126	135	144	152	161	170	179	188	9
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502	70070	079	088	096	105	114	122	131	140	148	9
503	157	165	174	183	191	200	209	217	226	234	9
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505	329	338	346	355	364	372	381	389	398	406	9
506	415	424	432	441	449	458	467	475	484	492	9
507	501	509	518	526	535	544	552	561	569	578	9
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514	096	105	113	122	130	139	147	155	164	172	8
515	181	189	198	206	214	223	231	240	248	257	8
516	265	273	282	290	299	307	315	324	332	341	8
517	349	357	366	374	383	391	399	408	416	425	8
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519	517	525	533	542	550	559	567	575	584	592	8
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522	767	775	784	792	800	809	817	825	834	842	8
523	850	858	867	875	883	892	900	908	917	925	8
524	933	941	950	958	966	975	983	991	999	*008	8
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526	099	107	115	123	132	140	148	156	165	173	8
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543	480	488	496	504	512	520	528	536	544	552	8
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552	194	202	210	218	225	233	241	249	257	265	8
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557	586	593	601	609	617	624	632	640	648	656	8
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572	740	747	755	762	770	778	785	793	800	808	8
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585	716	723	730	738	745	753	760	768	775	782	7
586	790	797	805	812	819	827	834	842	849	856	7
587	864	871	879	886	893	901	908	916	923	930	7
588	938	945	953	960	967	975	982	989	997	*004	7
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590	085	093	100	107	115	122	129	137	144	151	7
591	159	166	173	181	188	195	203	210	217	225	7
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604	104	111	118	125	132	140	147	154	161	168	7
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606	247	254	262	269	276	283	290	297	305	312	7
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608	390	398	405	412	419	426	433	440	447	455	7
609	462	469	476	483	490	497	504	512	519	526	7
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611	604	611	618	625	633	640	647	654	661	668	7
612	675	682	689	696	704	711	718	725	732	739	7
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615	888	895	902	909	916	923	930	937	944	951	7
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625	588	595	602	609	616	623	630	637	644	650	7
626	657	664	671	678	685	692	699	706	713	720	7
627	727	734	741	748	754	761	768	775	782	789	7
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635	277	284	291	298	305	312	318	325	332	339	7
636	346	353	359	366	373	380	387	393	400	407	7
637	414	421	428	434	441	448	455	462	468	475	7
638	482	489	496	502	509	516	523	530	536	543	7
639	550	557	564	570	577	584	591	598	604	611	7
640	618	625	632	638	645	652	659	665	672	679	7
641	686	693	699	706	713	720	726	733	740	747	7
642	754	760	767	774	781	787	794	801	808	814	7
643	821	828	835	841	848	855	862	868	875	882	7
644	889	895	902	909	916	922	929	936	943	949	7
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646	81023	030	037	043	050	057	064	070	077	084	7
647	090	097	104	111	117	124	131	137	144	151	7
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654	558	564	571	578	584	591	598	604	611	617	7
655	624	631	637	644	651	657	664	671	677	684	7
656	690	697	704	710	717	723	730	737	743	750	7
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667	413	419	426	432	439	445	452	458	465	471	7
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670	607	614	620	627	633	640	646	653	659	666	7
671	672	679	685	692	698	705	711	718	724	730	6
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673	802	808	814	821	827	834	840	847	853	860	6
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676	995	*001	*008	*014	*020	*027	*033	*040	*046	*052	6
677	83059	065	072	078	085	091	097	104	110	117	6
678	123	129	136	142	149	155	161	168	174	181	6
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683	442	448	455	461	467	474	480	487	493	499	6
684	506	512	518	525	531	537	544	550	556	563	6
685	569	575	582	588	594	601	607	613	620	626	6
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687	696	702	708	715	721	727	734	740	746	753	6
688	759	765	771	778	784	790	797	803	809	816	6
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690	885	891	897	904	910	916	923	929	935	942	6
691	948	954	960	967	973	979	985	992	998	*004	6
692	84011	017	023	029	036	042	048	055	061	067	6
693	073	080	086	092	098	105	111	117	123	130	6
994	136	142	148	155	161	167	173	180	186	192	6
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696	261	267	273	280	286	292	298	305	311	317	6
697	323	330	336	342	348	354	361	367	373	379	6
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703	696	702	708	714	720	726	733	739	745	751	6
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709	065	071	077	083	089	095	101	107	114	120	6
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714	370	376	382	388	394	400	406	412	418	425	6
715	431	437	443	449	455	461	467	473	479	485	6
716	491	497	503	509	516	522	528	534	540	546	6
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718	612	618	625	631	637	643	649	655	661	667	6
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726	094	100	106	112	118	124	130	136	141	147	6
727	153	159	165	171	177	183	189	195	201	207	6
728	213	219	225	231	237	243	249	255	261	267	6
729	273	279	285	291	297	303	308	314	320	326	6
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731	392	398	404	410	415	421	427	433	439	445	6
732	451	457	463	469	475	481	487	493	499	504	6
733	510	516	522	528	534	540	546	552	558	564	6
734	570	576	581	587	593	599	605	611	617	623	6
735	629	635	641	646	652	658	664	670	676	682	6
736	688	694	700	705	711	717	723	729	735	741	6
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738	806	812	817	823	829	835	841	847	853	859	6
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740	923	929	935	941	947	953	958	964	970	976	6
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746	274	280	286	291	297	303	309	315	320	326	6
747	332	338	344	349	355	361	367	373	379	384	6
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755	795	800	806	812	818	823	829	835	841	846	6
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765	366	372	377	383	389	395	400	406	412	417	6
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767	480	485	491	497	502	508	513	519	525	530	6
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777	89042	048	053	059	064	070	076	081	087	092	6
778	098	104	109	115	120	126	131	137	143	148	6
779	154	159	165	170	176	182	187	193	198	204	6
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825	645	651	656	661	666	672	677	682	687	693	5
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835	169	174	179	184	189	195	200	205	210	215	5
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840	428	433	438	443	449	454	459	464	469	474	5
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871	94002	007	012	017	022	027	032	037	042	047	5
872	052	057	062	067	072	077	082	086	091	096	5
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878	349	354	359	364	369	374	379	384	389	394	5
879	399	404	409	414	419	424	429	433	438	443	5
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883	596	601	606	611	616	621	626	630	635	640	5
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893	085	090	095	100	105	109	114	119	124	129	5
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896	231	236	240	245	250	255	260	265	270	274	5
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926	661	666	670	675	680	685	689	694	699	703	5
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928	755	759	764	769	774	778	783	788	792	797	5
929	802	806	811	816	820	825	830	834	839	844	5
930	848	853	858	862	867	872	876	881	886	890	5
931	895	900	904	909	914	918	923	928	932	937	5
932	942	946	951	956	960	965	970	974	979	984	5
933	988	993	997	*002	*007	*011	*016	*021	*025	*030	5
934	97035	039	044	049	053	058	063	067	072	077	5
935	081	086	090	095	100	104	109	114	118	123	5
936	128	132	137	142	146	151	155	160	165	169	5
937	174	179	183	188	192	197	202	206	211	216	5
938	220	225	230	234	239	243	248	253	257	262	5
939	267	271	276	280	285	290	294	299	304	308	5
940	313	317	322	327	331	336	340	345	350	354	5
941	359	364	368	373	377	382	387	391	396	400	5
942	405	410	414	419	424	428	433	437	442	447	5
943	451	456	460	465	470	474	479	483	488	493	5
944	497	502	506	511	516	520	525	529	534	539	5
945	543	548	552	557	562	566	571	575	580	585	5
946	589	594	598	603	607	612	617	621	626	630	5
947	635	640	644	649	653	658	663	667	672	676	5
948	681	685	690	695	699	704	708	713	717	722	5
949	727	731	736	740	745	749	754	759	763	768	5
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
950	97772	777	782	786	791	795	800	804	809	813	5
951	818	823	827	832	836	841	845	850	855	859	5
952	864	868	873	877	882	886	891	896	900	905	5
953	909	914	918	923	928	932	937	941	946	950	5
954	955	959	964	968	973	978	982	987	991	996	5
955	98000	005	009	014	019	023	028	032	037	041	5
956	046	050	055	059	064	068	073	078	082	087	5
957	091	096	100	105	109	114	118	123	127	132	5
958	137	141	146	150	155	159	164	168	173	177	5
959	182	186	191	195	200	204	209	214	218	223	5
960	227	232	236	241	245	250	254	259	263	268	5
961	272	277	281	286	290	295	299	304	308	313	5
962	318	322	327	331	336	340	345	349	354	358	5
963	363	367	372	376	381	385	390	394	399	403	5
964	408	412	417	421	426	430	435	439	444	448	5
965	453	457	462	466	471	475	480	484	489	493	4
966	498	502	507	511	516	520	525	529	534	538	4
967	543	547	552	556	561	565	570	574	579	583	4
968	588	592	597	601	605	610	614	619	623	628	4
969	632	637	641	646	650	655	659	664	668	673	4
970	677	682	686	691	695	700	704	709	713	717	4
971	722	726	731	735	740	744	749	753	758	762	4
972	767	771	776	780	784	789	793	798	802	807	4
973	811	816	820	825	829	834	838	843	847	851	4
974	856	860	865	869	874	878	883	887	892	896	4
975	900	905	909	914	918	923	927	932	936	941	4
976	945	949	954	958	963	967	972	976	981	985	4
977	989	994	998	*003	*007	*012	*016	*021	*025	*029	4
978	99034	038	043	047	052	056	061	065	069	074	4
979	078	083	087	092	096	100	105	109	114	118	4
980	123	127	131	136	140	145	149	154	158	162	4
981	167	171	176	180	185	189	193	198	202	207	4
982	211	216	220	224	229	233	238	242	247	251	4
983	255	260	264	269	273	277	282	286	291	295	4
984	300	304	308	313	317	322	326	330	335	339	4
985	344	348	352	357	361	366	370	374	379	383	4
986	388	392	396	401	405	410	414	419	423	427	4
987	432	436	441	445	449	454	458	463	467	471	4
988	476	480	484	489	493	498	502	506	511	515	4
989	520	524	528	533	537	542	546	550	555	559	4
990	564	568	572	577	581	585	590	594	599	603	4
991	607	612	616	621	625	629	634	638	642	647	4
992	651	656	660	664	669	673	977	682	686	691	4
993	695	699	704	708	712	717	721	726	730	734	4
994	739	743	747	752	756	760	765	769	774	778	4
995	782	787	791	795	800	804	808	813	817	822	4
996	826	830	835	839	843	848	852	856	861	865	4
997	870	874	878	883	887	891	896	900	904	909	4
998	913	917	922	926	930	935	939	944	948	952	4
999	957	961	965	970	974	978	983	987	991	996	4
N.	0	1	2	3	4	5	6	7	8	9	D.

23 skiddos
for yourself—
Shut up!
ditto
ring Ty

Miss Maud Shaffer.
and
Miss Mabel Moyer.
Hindsboro High School
1907-09.